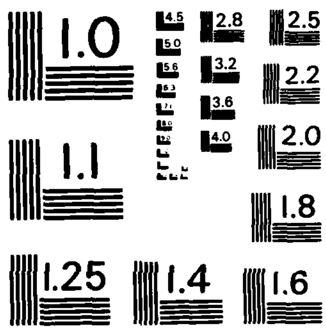
AD-A162 371 BUCKLING OF DELAMINATED SHELLS AND MULTI-ANNULAR PLATES 1/2
(U) GEORGIA INST OF TECH ATLANTA SCHOOL OF ENGINEERING
SCIENCE AN.. G J SIMITSES ET AL. OCT 85
UNCLASSIFIED AFOSR-TR-85-1081 AFOSR-83-0243 F/G 20/11 NL



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

AFOSR-TR-

AFOSR-TR- [illegible] 1081

AD-A162 371

BUCKLING OF DELAMINATED SHELLS AND MULTI-ANNULAR PLATES

by

George J. Simitses, Sayed Sallam
and Yeoshua Frostig

School of Engineering Science and Mechanics
GEORGIA INSTITUTE OF TECHNOLOGY
A Unit of the University System of Georgia
Atlanta, Georgia 30332

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER AFOSR-TR- 1081 2. GOVT ACCESSION NO. AD-A162 391	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) BUCKLING OF DELAMINATED SHELLS AND MULTI-ANNULAR PLATES	5. TYPE OF REPORT & PERIOD COVERED FINAL Sept. 1, 1984-August 31, 1985
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) George J. Simitses, S. Sallam, and Y. Frostig	8. CONTRACT OR GRANT NUMBER(s) AFOSR 83-0243
9. PERFORMING ORGANIZATION NAME AND ADDRESS Georgia Institute of Technology School of Engineering Science and Mechanics 225 North Avenue, S. W., Atlanta, GA 30332	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 2307/B1 PE 61102F
11. CONTROLLING OFFICE NAME AND ADDRESS AFOSR/NA Building 410 Bolling AFB, D. C. 20332	12. REPORT DATE October 1985
	13. NUMBER OF PAGES 161
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)	15. SECURITY CLASS. (of this report) Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Laminated shells; Delamination Buckling; Damage Tolerance; Annular Plates; Plates with Holes.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

~~The~~ buckling of delaminated thin cylindrical shells is investigated. The thin cylinders are axially-loaded and their geometry is virtually isotropic. The effects of boundary conditions, delamination size, and delamination position on the buckling load are studied. ~~Moreover, the~~ buckling analysis of multi-annular plates is presented. The annular sections are made of different materials, their size is varied, and the composite circular plates are supported in various ways. The rigidities of the various parts are varied in order to cover the cases of plates with holes, with rigid inclusion, and ring stiffeners.

DD FORM 1 JAN 73 1473   EDITION OF 1 NOV 65 IS OBSOLETE

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

BUCKLING OF DELAMINATED SHELLS AND MULTI-ANNULAR PLATES*

by

George J. Simitses+, Sayed Sallam++, and Yeoshua Frostig+++

School of Engineering Science and Mechanics
Georgia Institute of Technology, Atlanta, GA.

---

* This work was supported by the United States Air Force Office of Scientific Research under Grant AFOSR-83-0243.

+ Professor and Principal Investigator

++ Graduate Research Assistant

+++ Postdoctoral Fellow

SUMMARY

This report summarizes the second year work under the general heading of "Effect of Local Material Imperfections on the Buckling Behavior of Composite Structural Elements".

It describes two important areas: one of delamination buckling of cylindrical shells under axial compression, and second buckling of multi-annular plates, which emphasizes the effect of holes and foreign rigid inclusions.

The findings of the research are reported in two parts. The first part (Part A) deals with buckling of axially-loaded delaminated thin cylindrical shells. The geometry is assumed to be virtually isotropic and the size and position of the delamination, which extends around the entire circumference of the cylinder, are varied. Moreover, two sets of boundary conditions are considered simply supported and clamped.

The second part deals with buckling of multi-annular plates, made out of different materials, supported in various ways, and subjected to radial compressive loads. The rigidities of the parts are varied in order for the analysis to include plates with holes, rigid inclusions, and ring stiffeners.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

The use of fiber-reinforced composite laminates is rapidly increasing due to their excellent undirectional properties and their high strength to weight ratio. Recently, composite laminates have been used in space vehicles in the form of circular cylindrical shells, as a primary load carrying structure. In order to use laminated composite structures properly and effectively, it is important to predict and understand their behavior under the different external actions.

Imperfections have a very significant effect on the structural response and on its load carrying capacity. In addition to the usual geometric imperfections that are inherent in metallic plates and shells, laminated geometries are prone to such defects as broken fibers, cracks in the matrix material, delaminated areas (separatioin of adjoining plies), as well as holes and small voids.

Delamination is one of the most common failure modes in composite materials. Delamination is developed as a result of imperfections in the production technology or due to the effect of certain factors during the operational life of the laminate, such as impact by foreign objects. The presence of delamination in a laminated composite material may cause local buckling and reduction in the overall stiffness of the structure, which may lead to early failure.

The problems of delaminated composite structures have received attention in recent years. As far as the buckling response of delaminated structures is concerned, an extensive review is given in Ref.[1].

The problem of buckling of delaminated cylindrical shells has not received the deserved attention. Very few investigations have been carried out in this area. Kulkarni and Frederick [2] use a "branched integration" technique to solve the problem of buckling of a two-layered cylindrical shell,partially debonded, and subjected to axial compression. They [2] consider the case where the delamination originated at the boundary. Results are reported [2] for different lengths of debonding and inner to outer layer thickness ratios. A significant decrease in the critical load is observed. The buckling of stiffened circular cylindrical shells, with two unbonded orthotropic layers, is reported by Jones [3]. Jones [3] assumes that the layers do not seperate during buckling, i.e. the deformation of both layers are assumed to be the same. He also examines the case when one of the two unbonded orthotropic layers is circumferentially cracked. His results for a cylindrical shell made of aluminum with ablative outer layer and subjected to hydrostatic pressure show that the ablative layer had to be increased by 50% in thickness in the damaged (debonded) cylindrical shell in order to obtain the same buckling load as that of the perfect cylindrical shell. Troshin [4] examines the effect of longitudinal delamination in a laminar cylindrical shell on the critical external pressure. The delamination is assumed to extend over the entire length of the shell. He [4], reports on the effect of the length and position

of the delamination on the critical external pressure.

In the present research the study of delaminated thin cylindrical shells of perfect geometry and under uniform axial compression is presented.

The buckling equations are derived by employing a perturbation technique. The solution is of the separated form with periodic functions in the circumferential coordinate (sine or cosine), and complex exponential functions in the axial coordinate.

CHAPTER II

BUCKLING OF DELAMINATED CYLINDRICAL SHELLS

2.1 MATHEMATICAL FORMULATION

The effect of circumferential delamination on the critical loads, for a laminated circular cylindrical shell and subjected to compressive axial loading, is studied. The material behavior of the cylindrical shell is assumed to be linearly elastic. Delamination is assumed to exist before the load is applied, and extends along the entire circumference of the cylindrical shell, on a surface parallel to the reference surface (see Figure 1). The location and size of the delamination is arbitrary and the boundary conditions are either simply supported or clamped. Delamination separates the cylindrical shell into four regions (four thin-walled cylindrical shells), such that each region is symmetric with respect to its own mid-surface.

The governing equations for the buckling of the delaminated cylindrical shell are derived based on the thin shell theory (Kirchhoff-Love hypotheses), Donell-type of kinematic relations, and the assumption of existence of a membrane state, before buckling (classical approach).

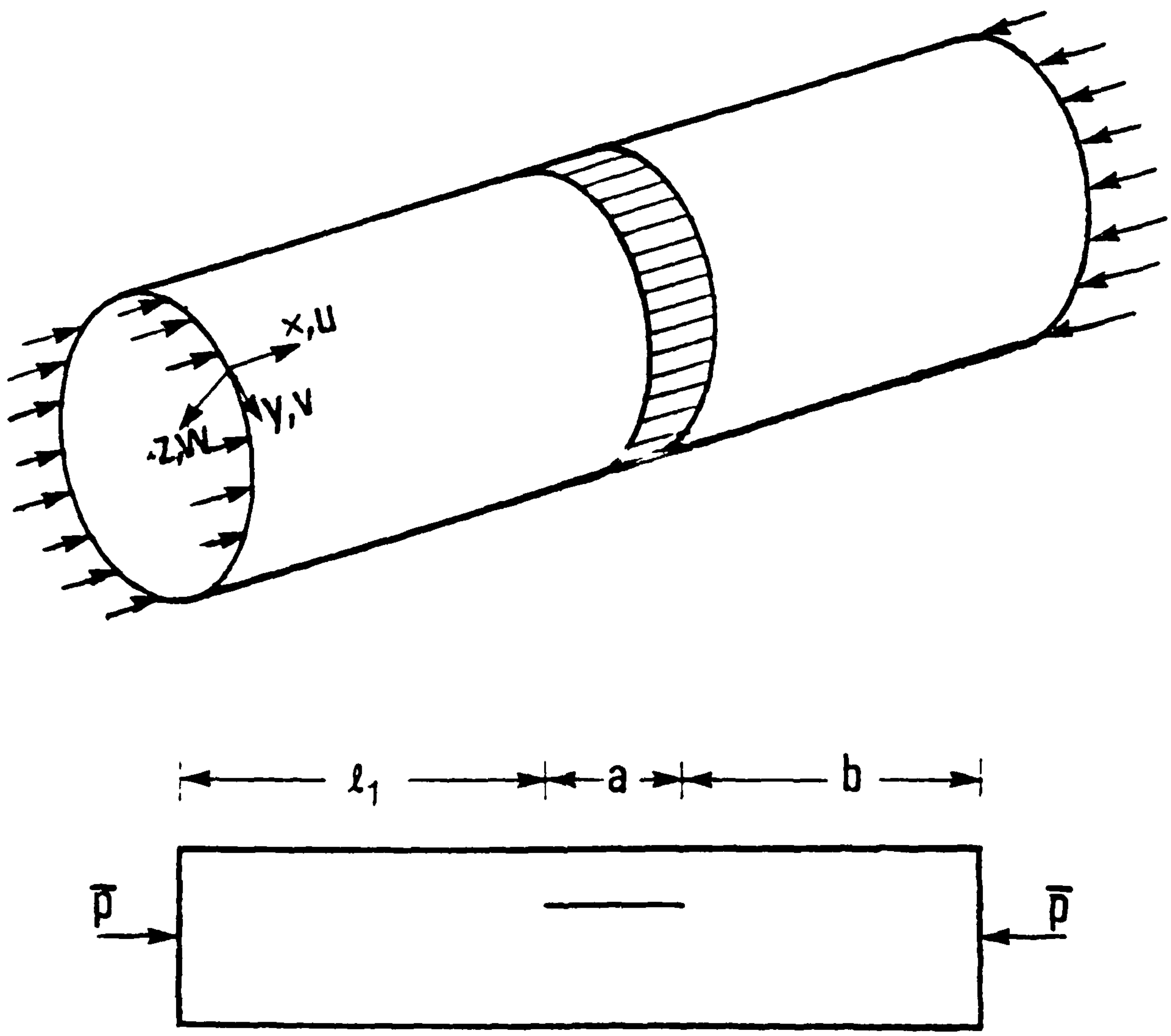


Figure 1 Geometry and Loading

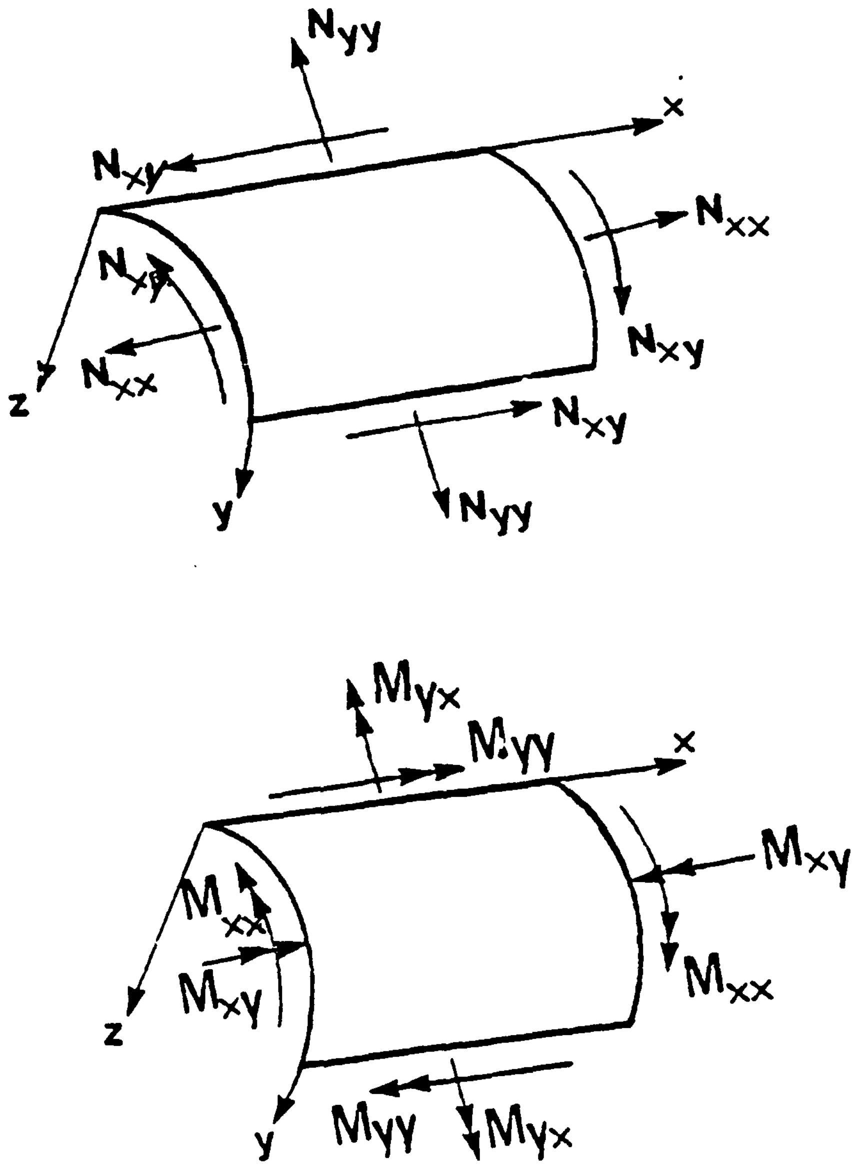


Figure 2 Sign Convention

The geometry and sign convention are shown in figures 1 and 2 . The coordinate system is such that x is measured from the left end. Moreover u_i , v_i and w_i (i=1,2,3,4) denote the axial, circumferential and radial displacements of the material points on the mid surface of each region, respectively .

2.1.1 Kinematic Relations

The axial and circumferential displacements of any material point (x, z_i) are given by

$$u_i(x,y,z_i) = u_i^o(x,y) - z_i\, w_{i,x}$$
$$v_i(x,y,z_i) = v_i^o(x,y) - z_i\, w_{i,y} \tag{1}$$

where u^o and v^o are components of "middle surface" material points , z_i is measured from the mid surface of each region and the comma denotes partial differentiation with respect to the index that follows.

The kinematic (Donnell-type) relations are given by :

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix}_i = \begin{bmatrix} \varepsilon^o_{xx} \\ \varepsilon^o_{yy} \\ \varepsilon^o_{xy} \end{bmatrix}_i - z_i \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ 2\kappa_{xy} \end{bmatrix}_i \tag{2}$$

where the superscript "o" denotes reference strains and the k's denote the reference surface changes in curvature and torsion.

Note that for each region the reference surface has been placed at the middle surface . The reference surface strains are defined in terms of the displacements as:

$$
\begin{aligned}
\epsilon^{o}_{xx_i} &= u_{i,x} + w^{2}_{i,x} \\
\epsilon^{o}_{yy_i} &= v_{i,y} - w_i/R + \tfrac{1}{2}w^{2}_{i,y} \\
\gamma^{o}_{xy_i} &= u_{i,y} + v_{i,x} + w_{i,x}\, w_{i,y}
\end{aligned}
\tag{3}
$$

and the changes in curvature and torsion are given in terms of the transverse displacement w ,by:

$$
\begin{aligned}
\kappa_{xx_i} &= w_{i,xx} \\
\kappa_{yy_i} &= w_{i,yy} \\
\kappa_{xy_i} &= w_{i,xy}
\end{aligned}
\tag{4}
$$

8

2.1.2 Stress-Strain Relations

Each lamina of the composite shell can be considered as orthotropic with two principal material directions (1 and 2) parallel and perpendicular to the direction of the filaments.

The stress-strain relations for each lamina are:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & 2Q_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix}$$

The relation between the elements of the stiffness matrix [Q] and the engineering material constants (E_{11} , E_{22} , G_{12} , ν_{12}) ,are :

$$\begin{aligned} Q_{11} &= E_{11}/\,(1-\nu_{12}\nu_{21}) \quad ; \quad Q_{12} = \nu_{21}E_{11}\,/\,(1-\nu_{12}\nu_{21}) \\ Q_{22} &= E_{22}/\,(1-\nu_{12}\nu_{21}) \quad ; \quad Q_{33} = G_{12} \end{aligned} \tag{5}$$

For a general lamina (k) with fibers making an angle θ_k with the x-axis then,

$$\{\sigma\}_i^k = [\bar{Q}]_i^k \{\epsilon\}_i^k$$

where

$$\{\sigma\} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} \qquad \{\varepsilon\} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix} \tag{6}$$

and

$$[\bar{Q}] = [T]^{-1} [Q] [T] \tag{7}$$

where [T] is the transformation matrix [5] .

It is more convenient to deal with stress and moment resultants rather than dealing with stresses in deriving the governing equations. The stress and moment resultants are defined by

$$\begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{bmatrix}_i = \int_{h_o}^{h_n} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}_i dz \tag{8}$$

$$
\begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix}_i = \int_{h_o}^{h_n} z \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}_i dz \tag{9}
$$

The stress and moment resultants are related to the reference surface (for each region) strains and changes in curvature and torsion by

$$
\begin{Bmatrix} N \\ M \end{Bmatrix}_I = \begin{bmatrix} A & -B \\ B & -D \end{bmatrix}_I \begin{Bmatrix} \varepsilon^o \\ K \end{Bmatrix}_I \tag{10}
$$

$$I = 1,2,3,4$$

where

$$
A_{ij_I} = \sum_{k=1}^{N} \overline{Q}^{(k)}_{ij_I} (h_k - h_{k-1})
$$

$$
B_{ij_I} = \frac{1}{2} \sum_{k=1}^{N} \overline{Q}^{(k)}_{ij_I} (h_k^2 - h_{k-1}^2) \tag{11}
$$

$$
D_{ij_I} = \frac{1}{3} \sum_{k=1}^{N} \overline{Q}^{(k)}_{ij_I} (h_k^3 - h_{k-1}^3)
$$

$$I=1,2,3,4$$

In deriving the governing equations, it is assumed that each region of the delaminated cylindrical shell is symmetric about its mid-surface hence, there is no coupling between bending and stretching, $B_{ij} = 0$.

Moreover , only cases where the fibers are in the axial direction and/or in the circumferential direction are considered, which means that the stiffnesses $A_{13}, A_{23}, D_{13}, D_{23}$ are zero. Thus , Eqs(10) assume the form

$$\begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{bmatrix} = \begin{bmatrix} A_{xx} & A_{xy} & 0 \\ A_{xy} & A_{yy} & 0 \\ 0 & 0 & A_{ss} \end{bmatrix} \begin{bmatrix} \epsilon^o_{xx} \\ \epsilon^o_{yy} \\ \gamma^o_{xy} \end{bmatrix} \qquad (12\text{-a})$$

$$\begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} = \begin{bmatrix} D_{xx} & D_{xy} & 0 \\ D_{xy} & D_{yy} & 0 \\ 0 & 0 & D_{ss} \end{bmatrix} \begin{bmatrix} -w_{,xx} \\ -w_{,yy} \\ -w_{,xy} \end{bmatrix} \qquad (12\text{-b})$$

2.1.3 Equilibrium Equations

The equilibrium equations for a thin cylindrical shell are listed below

$$N_{xx_{i,x}} + N_{xy_{i,y}} = 0 \qquad (13)$$

$$N_{xy_{i,x}} + N_{yy_{i,y}} = 0 \ . \qquad (14)$$

$$M_{xx_{i,xx}} + 2\, M_{xy_{i,xy}} + M_{yy_{i,yy}} + N_{yy_i}/R + N_{xx_i} w_{i,xx} +$$

$$+ 2\, N_{xy_i} w_{i,xy} + N_{yy_i} w_{i,yy} = 0 \qquad (15)$$

2.1.4 Boundary and Auxiliary Conditions

Boundary Conditions (at $x = 0 , L$)

$$
\begin{array}{lll}
\text{Either} & \text{Or} & \\
N_{xx} = \bar{N}_{xx} & u = \bar{u} & \\
N_{xy} = \bar{N}_{xy} & v = \bar{v} & (16) \\
N_{xx}w_{,x}+N_{xy}w_{,y}+2M_{xy,y}+M_{xx,x}=\bar{Q}_x & w = \bar{w} & \\
M_{xx} = \bar{M}_{xx} & w_{,x} = \bar{w}_{,x} &
\end{array}
$$

Kinematic Continuity Conditions

at $x = \ell$

$$\left.\begin{aligned} u_1 - \frac{h}{2} w_{1,x} &= u_2 \\ u_2 + \frac{H}{2} w_{1,x} &= u_3 \end{aligned}\right\} \tag{17}$$

$$\left.\begin{aligned} v_1 - \frac{h}{2} w_{1,y} &= v_2 \\ v_1 + \frac{H}{2} w_{1,y} &= v_3 \end{aligned}\right\} \tag{18}$$

$$w_1 = w_2 = w_3 \tag{19}$$

$$w_{1,x} = w_{2,x} = w_{3,x} \tag{20}$$

$x = \ell + a$

$$\left.\begin{aligned} u_4 - \frac{h}{2} w_{4,x} &= u_2 \\ u_4 + \frac{H}{2} w_{4,x} &= u_3 \end{aligned}\right\} \tag{21}$$

$$\left.\begin{aligned} v_4 - \frac{h}{2} w_{4,y} &= v_2 \\ v_4 + \frac{H}{2} w_{4,y} &= v_3 \end{aligned}\right\} \tag{22}$$

$$w_4 = w_2 = w_3 \tag{23}$$

$$w_{4,x} = w_{2,x} = w_{3,x} \tag{24}$$

Continuity in Moments and Forces

at $x = \ell$

$$M_{xx_1} - M_{xx_2} - M_{xx_3} - N_{xx_2} \frac{h}{2} + N_{xx_3} \frac{H}{2} = 0 \tag{25}$$

$$N_{xx_1} - N_{xx_2} - N_{xx_3} = 0 \tag{26}$$

$$N_{xy_1} - N_{xy_2} - N_{xy_3} = 0 \tag{27}$$

$$Q_{x_1} - Q_{x_2} - Q_{x_3} = 0 \tag{28}$$

at $x = \ell + a$

$$M_{xx_4} - M_{xx_2} - M_{xx_3} - N_{xx_2} \frac{h}{2} + N_{xx_3} \frac{H}{2} = 0 \tag{29}$$

$$N_{xx_4} - N_{xx_2} - N_{xx_3} = 0 \tag{30}$$

$$N_{xy_4} - N_{xy_2} - N_{xy_3} = 0 \tag{31}$$

$$Q_{x_4} - Q_{x_2} - Q_{x_3} = 0 \tag{32}$$

wher

$$Q_{x_i} = M_{xx_{i,x}} + 2\, M_{xy_{i,y}} - N_{xx_i}\, w_{i,x}$$

$$i=1,2,3,4$$

2.2 PRIMARY (MEMBRANE) STATE SOLUTION

As the compressive load $\bar{p}$ is applied (quasi) statically,a (membrane) primary state exists. Assume the initial curvature to be the same for all the four regions i.e. $t \ll R$. Also,assume that a uniform radial displacement w_i is allowed. Then , the primary state is characterized by:

$$M^p_{xx_i} = M^p_{yy_i} = M^p_{xy_i} = N^p_{yy_i} = N^p_{xy_i} = Q^p_{x_i} = 0 \tag{33}$$

$$N^p_{xx_i} = p_i \tag{34}$$

where

$$p_1 = p_4 = -\bar{p} \quad ; \quad p_2 = -\bar{H}\,\bar{p} \quad ; \quad p_3 = -\bar{h}\,\bar{p}$$

where $\bar{p}$ is the compressive applied load ,and $\bar{h}=h/t$ and $\bar{H}=H/t$

Furthermore,

$$u_i^p = -\bar{p}_i/C_{xx_i} \quad x \qquad ; \qquad v_i^p = 0$$

$$w_i^p = -(\bar{p}_i/C_{xx_i})\ (A_{xy}/A_{yy})_i \qquad\qquad i=1,2,3,4 \tag{35}$$

where
$$C_{xx_i} = (A_{xx} - A_{xy}^2/A_{yy})_i$$

2.3 BUCKLING EQUATIONS

The buckling equations are derived by employing a perturbation technique. Since the perturbed state can be chosen to be infinitessimally close to the primary state only , first order terms in the admissible variations are retained.

The following expressions are substituted in the governing equations

$$u_i = u_i^p + u_i^a \quad ; \quad v_i = v_i^a \quad ; \quad w_i = w_i^p + w_i^a$$

$$N_{xx_i} = N_{xx_i}^p + N_{xx_i}^a \quad ; \quad N_{xy_i} = N_{xy_i}^a \quad ; \quad N_{yy_i} = N_{yy_i}^a$$

$$M_{xx_i} = M_{xx_i}^a \quad ; \quad M_{xy_i} = M_{xy_i}^a \quad ; \quad M_{yy_i} = M_{yy_i}^a$$

$$Q_{x_i} = Q_{x_i}^a$$

where the super "a" parameters are referred to the additional changes(that corespond to admissible variations u_i^a ,v_i^a and w_i^a).

By retaining only linear terms in the additional paramters, the buckling equations are:

$$N^a_{xx_{i,x}} + N^a_{xy_{i,y}} = 0 \tag{36}$$

$$N^a_{xy_{i,x}} + N^a_{yy_{i,y}} = 0\ . \tag{37}$$

$$M^a_{xx_{i,xx}} + 2\, M^a_{xy_{i,xy}} + M^a_{yy_{i,yy}} + N^a_{yy_i}/R - \bar{N}_{xx_i}\, w^a_{i,xx} = 0 \tag{38}$$

In terms of the in-plane and transverse displacement(u,v,w) , the buckling equations take the form

$$D_{xx_i}\, w^a_{i,xxxx} + 2(D_{xy}+2D_{ss})_i\, w^a_{i,xxyy} + D_{yy_i}\, w^a_{i,yyyy}$$
$$+ \bar{N}_{xx_i}\, w^a_{i,xx} - \{ A_{xy_i}\, u^a_{i,x} + A_{yy_i}\, (v^a_{i,y} - w^a_i/R)\}/R = 0 \tag{39}$$

$$L_{1_i}\, u^a_i + L_{2_i}\, v^a_i = A_{xy_i}\, w^a_{i,x}/R \tag{40}$$

$$L_{2_i}\, u^a_i + L_{3_i}\, v^a_i = A_{yy_i}\, w^a_{i,y}/R \tag{41}$$

where the differential operators L_1, L_2 and L_3 are given by

$$L_1 = A_{xx}\,\partial^2/\partial x^2 + A_{ss}\,\partial^2/\partial y^2$$

$$L_2 = (A_{xy}+A_{ss})\,\partial^2/\partial x\partial y$$

$$L_3 = A_{ss}\,\partial^2/\partial x^2 + A_{yy}\,\partial^2/\partial y^2$$

Equations (40),(41) can also be written as:

$$L_i\, u_i^a = A_{ss_i}\,[\,A_{xy_i}\,\partial^3/\partial x^3 - A_{yy_i}\,\partial^3/\partial x\partial y^2\,]\; w_i^a/R$$

i.e.

$$u_i^a = L_i^{-1}\{A_{ss_i}[A_{xy_i}\,\partial^3/\partial x^3 - A_{yy_i}\,\partial^3/\partial x\partial y^2]\; w_i^a/R\} \qquad (42)$$

$$L_i\, v_i^a = [A_{yy_i}A_{ss_i}\partial^3/\partial y^3 - (A_{xy_i}^2 + A_{xy_i}A_{ss_i} - A_{xx_i}A_{yy_i})\partial^3/\partial x^2\partial y]\; w_i^a/R$$

i.e.

$$v_i^a = L_i^{-1}\{[A_{yy_i}A_{ss_i}\partial^3/\partial y^3 - (A_{xy_i}^2 + A_{xy_i}A_{ss_i} - A_{xx_i}A_{yy_i})\partial^3/\partial x^2\partial y]\; w_i^a/R\} \qquad (43)$$

where

$$\overline{L_i} = A_{xx_i} A_{ss_i} \partial^4/\partial x^4 - (A_{xy}^2 + 2A_{xy}A_{ss} - A_{xx}Ayy)_i \partial^4/\partial x^2 \partial y^2 + A_{yy_i} A_{ss_i} \partial^4/\partial y^4$$

$$\overline{L_i}\, \overline{L_i}^{-1} = 1$$

Equations (42) and (43) are used to eliminate u_i and v_i from Eq.(39) to obtain one (buckling) equation with higher order derivatives in the transverse deflection w_i .

Thus, the buckling equation for each region , i , becomes

$$\begin{aligned}
&[\{A_{xx_i} A_{ss_i} \partial^4/\partial x^4 - (A_{xy}^2 + 2A_{xy}A_{ss} - A_{xx}A_{yy})_i \partial^4/\partial x^2 \partial y^2 + A_{yy_i} A_{ss_i} \partial^4/\partial y^4\} \\
&X \{D_{xx_i} \partial^4/\partial x^4 + 2(D_{xy} + 2D_{ss})_i \partial^4/\partial x^2 \partial y^2 + D_{yy_i} \partial^4/\partial y^4 + N_{xx_i} \partial^2/\partial x^2\} \\
&\qquad + A_{ss_i} A_{xx_i} \{A_{yy} - A_{xy}^2/A_{xx}\}_i \partial^4/\partial x^4]\, w_i^a/R^2 = 0 \qquad (44)
\end{aligned}$$

$$i=1,2,3,4$$

The related conditions are grouped into three categories : Boundary Conditions , Kinematic Continuity Conditions and Continuity in Moments and Forces.

Boundary Conditions:

Along each end of the cylindrical shell four boundary conditions have to be satisfied. For both simply supported and clamped boundaries, the following possibilities hold (Ref 6) :

a) Simply Supported

$$\begin{array}{ll} \text{SS1} & w_j^a = w_{j,xx}^a = N_{xx_j}^a = N_{xy_j}^a = 0 \\ \text{SS2} & w_j^a = w_{j,xx}^a = u_j^a = N_{xy_j}^a = 0 \\ \text{SS3} & w_j^a = w_{j,xx}^a = N_{xx_j}^a = v_j^a = 0 \\ \text{SS4} & w_j^a = w_{j,xx}^a = u_j^a = v_j^a = 0 \end{array} \tag{45}$$

b) Clamped Support

$$\begin{array}{ll} \text{CC1} & w_j^a = w_{j,x}^a = N_{xx_j}^a = N_{xy_j}^a = 0 \\ \text{CC2} & w_j^a = w_{j,x}^a = u_j^a = N_{xy_j}^a = 0 \\ \text{CC3} & w_j^a = w_{j,x}^a = N_{xx_j}^a = v_j^a = 0 \\ \text{CC4} & w_j^a = w_{j,x}^a = u_j^a = v_j^a = 0 \end{array} \tag{46}$$

where $j=1$ at $x=0$ and $j=4$ at $x=L$

Kinematic Continuity Conditions

at $x = \ell$

$$\left.\begin{aligned} u_1^a - \frac{h}{2} w_{1,x}^a &= u_2^a \\ u_2^a + \frac{H}{2} w_{1,x}^a &= u_3^a \end{aligned}\right\} \tag{47}$$

$$\left.\begin{aligned} v_1^a - \frac{h}{2} w_{1,y}^a &= v_2^a \\ v_1^a + \frac{H}{2} w_{1,y}^a &= v_3^a \end{aligned}\right\} \tag{48}$$

$$w_1^a = w_2^a = w_3^a \tag{49}$$

$$w_{1,x}^a = w_{2,x}^a = w_{3,x}^a \tag{50}$$

at $x = \ell + a$

$$\left.\begin{aligned} u_4^a - \frac{h}{2} w_{4,x}^a &= u_2^a \\ u_4^a + \frac{H}{2} w_{4,x}^a &= u_3^a \end{aligned}\right\} \tag{51}$$

$$\left.\begin{aligned} v_4^a - \frac{h}{2} w_{4,y}^a &= v_2^a \\ v_4^a + \frac{H}{2} w_{4,y}^a &= v_3^a \end{aligned}\right\} \tag{52}$$

$$w_4^a = w_2^a = w_3^a \tag{53}$$

$$w_{4,x}^a = w_{2,x}^a = w_{3,x}^a \tag{54}$$

Continuity in Moment and Forces

$x = \ell$

$$M^a_{xx_1} - M^a_{xx_2} - M^a_{xx_3} - N^a_{xx_2} \frac{h}{2} + N^a_{xx_3} \frac{H}{2} = 0 \qquad (55)$$

$$N^a_{xx_1} - N^a_{xx_2} - N^a_{xx_3} = 0 \qquad (56)$$

$$N^a_{xy_1} - N^a_{xy_2} - N^a_{xy_3} = 0 \qquad (57)$$

$$Q^a_{x_1} - Q^a_{x_2} - Q^a_{x_3} = 0 \qquad (58)$$

at $x = \ell + a$

$$M^a_{xx_4} - M^a_{xx_2} - M^a_{xx_3} - N^a_{xx_2} \frac{h}{2} + N^a_{xx_3} \frac{H}{2} = 0 \qquad (59)$$

$$N^a_{xx_4} - N^a_{xx_2} - N^a_{xx_3} = 0 \qquad (60)$$

$$N^a_{xy_4} - N^a_{xy_2} - N^a_{xy_3} = 0 \qquad (61)$$

$$Q^a_{x_4} - Q^a_{x_2} - Q^a_{x_3} = 0 \qquad (62)$$

2.4 SOLUTION OF BUCKLING EQUATIONS

The solution of the buckling equations ,Eq.(44) ,can be written as [6] :

$$w_i = \sum_{j=1}^{8} A_{ij} \, e^{\lambda_{ij} \frac{x}{R} \left(\frac{2E_{11}}{\sigma_{c\ell}}\right)_i^{1/2}} \sin S_i \frac{y}{R}\left(\frac{2E_{11}}{\sigma_{c\ell}}\right)_i^{1/2} \tag{63}$$

where

$$\sigma_{c\ell_i} = \frac{E_{11}}{\sqrt{3(1 - \nu_{12}\,\nu_{21})}} \frac{t_i}{R}$$

λ_{ij} are the characteristic roots of the buckling equation, and can be found by solving the following equation:

$$\left[\lambda_{ij}^4 + \left(\frac{A_{yy}}{A_{xx}}\right)_i S_i^4 + \left(\frac{A_{xy}^2 + 2A_{xy}\,A_{ss} - A_{xx}\,A_{yy}}{A_{xx}\,A_{ss}}\right)_i \lambda_{ij}^2\, S_i^2\right]$$

$$+ \left[\lambda_{ij}^4 + \left(\frac{D_{yy}}{D_{xx}}\right)_i S_i^4 - 2\left(\frac{D_{xy} + 2D_{ss}}{D_{xx}}\right)_i \lambda_{ij}^2\, S_i^2 + 2\,\rho_i \frac{1}{g_{x_i}} \lambda_{ij}^2\right]$$

$$+ \frac{1}{t_i\, E_{11}\, g_{x_i}} \left(A_{yy} - \frac{A_{xy}^2}{A_{xx}}\right)_i \lambda_{ij}^4 = 0 \tag{64}$$

$$i = 1, 2, 3, 4$$

where

$$\rho_i = \frac{N_{xx_i}}{t_i\, \sigma_{c\ell_i}} \quad ; \quad g_{x_i} = \frac{D_{xx_i}}{\frac{E_{11}\, t_i^3}{12(1-\nu_{12}\nu_{21})}}$$

Note that for continuity of displacement around the circumferential direction

$$S_i\left(\frac{2\ E_{11}}{\sigma_{c\ell}}\right)_i^{1/2} = n$$

where n is the number of full waves in the circumferential direction. Also note that n must be the same for all regions ,for continuity of displacements to be satisfied .

Using Eqs(42),(43) & (63) , the displacements , stress and moment resultants are expressed in terms of λ_{ij} , S_i and A_{ij} as follows:

$$u_i = \sum_{j=1}^{8} \left(\frac{\sigma_{c\ell}}{2E_{11}}\right)_i^{1/2} \frac{\left[\left(\frac{A_{yy}}{A_{xx_i}}\ \lambda_{ij}^2 + \left(\frac{A_{yy}}{A_{xx}}\right)_i S_i^2\right]\ \lambda_{ij}}{f\ (\lambda_{ij})}\ W_s \tag{65}$$

$$V_i = -\sum_{j=1}^{8} \left(\frac{\sigma_{c\ell}}{2\ E_{11}}\right)_i^{1/2} \frac{\left[\left(\frac{A_{xy}^2 + A_{xy}\ A_{ss} - A_{xx}\ A_{yy}}{A_{xx}\ A_{ss}}\right)_i \lambda_{ij}^2 + \left(\frac{A_{yy}}{A_{xx}}\right)_i S_i^2\right] S_i}{f(\lambda_{ij})}\ W_c \tag{66}$$

where

$$f(\lambda_{ij}) = \lambda_{ij}^4 + \left(\frac{A_{yy}}{A_{xx}}\right)_i S_i^4 + \left(\frac{A_{xy}^2 + 2A_{xy}\ A_{ss} - A_{xx}\ A_{yy}}{A_{xx}\ A_{ss}}\right)_i \lambda_{ij}^2\ S_i^2$$

$$W_s = A_{ij} \, e^{\lambda_{ij} \frac{x}{R} \left(\frac{2 E_{11}}{\sigma_{c\ell}}\right)_i^{1/2}} \sin \left[n \frac{y}{R}\right]$$

$$W_c = A_{ij} \, e^{\lambda_{ij} \frac{x}{R} \left(\frac{2E_{11}}{\sigma_{c\ell}}\right)_i^{1/2}} \cos \left(n \frac{y}{R}\right)$$

$$N_{xx_i} = \frac{1}{R} \sum_{j=1}^{8} \frac{\left(A_{yy} - \frac{A_{xy}^2}{A_{xx}}\right)_i}{f(\lambda_{ij})} \lambda_{ij}^2 \, S_i^2 \, W_s \tag{67}$$

$$N_{xy_i} = \frac{1}{R} \sum_{j=1}^{8} \frac{\left(A_{yy} - \frac{A_{xy}^2}{A_{xx}}\right)_i}{f(\lambda_{ij})} \lambda_{ij}^3 \, S_i \, W_c \tag{68}$$

$$M_{xx_i} = \frac{D_{xx_i}}{R} \left(\frac{2E_{11}}{\sigma_{c\ell}}\right)_i \sum_{j=1}^{8} \left[\left(\frac{D_{xy}}{D_{xx}}\right)_i S_i^2 - \lambda_{ij}^2\right] W_s \tag{69}$$

$$Q_i = - \frac{D_{xx_i}}{R} \left(\frac{\sigma_{c\ell}}{2 E_{11}}\right)_i^{3/2} \sum_{j=1}^{8} \lambda_{ij} \left[\lambda_{ij}^2 - \left(\frac{D_{xy} + 2 D_{ss}}{D_{xx}}\right)_i S_i^2 + 2 \frac{\rho_i}{g_{x_i}}\right] W_s \tag{70}$$

The solution to the buckling equations, Eqs (63), requires knowledge of 32 constant (A_{ij}, i=1,2,3,4 ,j=1,2,3,...8). There exist 32 boundary and continuity conditions, which are homogeneous in u_i , v_i and w_i and their space derivatives. These consist of eight boundary conditions, four at each end, Eqs (45) or Eqs (46), sixteen kinematic continuity conditions, Eqs (47) - (54), and eight continuity conditions in moments and forces, Eqs (55) - (62).

The use of the above system of boundary and continuity conditions yields a system of linear, homogeneous, algebraic equations in A_{ij} . The resulting system of equations takes the form

$$[C]\{X\} = [0] \qquad (71)$$

where [C] is a 32x32 matrix and {X} an 1x32 (column) matrix. The elements of matrix C are given in appendix A. For a nontrivial solution to exist, the determinant of the coefficients must vanish. The determinant contains geometric parameters (L,R,a,h), material parameters (E_{11}, E_{22}, G_{12}, ν_{12}), the applied load, $\bar{p}$, and the circumferential wave number, n.

For a given geometry and material properties, the circumferential wave number (n) is varied and the corresponding load which makes the determinant vanish is obtained. The lowest of these loads is the critical load.

2.5 RESULTS AND DISCUSSION

Buckling analysis of delaminated cylindrical shells involves many geometric and material parameters. Also, as is well known, the different inplane boundary conditions have a significant effect on the critical load of the cylindrical shells. Therefore, general results, which account for all the effect of the different parameters on the critical load of the delaminated cylindrical shell, require several extended studies. Accordingly, the results presented herein are restricted to specific numerical examples in order to illustrate the applicability of the present model.

In generating results for the present model, it is assumed that there is no contact between the two regions across the delamination at the instant of buckling. Otherwise the problem has to be treated as if a cylindrical shell is resting on an elastic foundation (not attached to the foundation). A model for accounting for thin behavior is suggested in Appendix B.

For a given configuration, the critical load, $\bar{p}$, is determined as the smallest eigenvalue that makes the determinant of the coefficients of the system of homogeneous algebraic equations (see Eq.71) equal to zero. The circumferential wave number is varied in order to determine the smallest of the eignvalues.

Results are generated for a cylindrical shell made up of isotropic laminate, and for both simply supported and clamped boundary conditions. Since for each type of supports (simple and clamped supports) there are four types of boundary conditions, results are generated for the weakest

and for the strongest configurations, which are known as SS1 ($w = w_{,xx} = \sigma_{xx} = \tau_{xy} = 0$) and CC4 ($w = w_{,x} = u = v = 0$), respectively. The boundary conditions are assumed to be the same at both ends, i.e. both ends are SS1 or CC4. The dimenstions of the cylindrical shell are such that L/R=5, and R/t=30, where L, R and t are the length, radius and thickness of the shell, respectively.

Figure 3 shows the effect of delaminatiand length and z-position on the critical loads of a simply supported cylindrical shell (SS1). The delamination is assumed to be located symmetrically with respect to both ends of the shell. The critical loads are normalized with respect to the critical load of the classical theory. The figure shows that for a delamination in the middle surface of the cylindrical shell, the delamination has a negligible effect on the critical loads as long as the delamination is far from the edges (effect of position with respect to the edges will be discussed in another figure). As the delamination moves away from the middle surface of the shell, its presence becomes important. For a delamiation thickness $\bar{h} = 0.3$, the delamination has no effect on the critical loads, as long as the delamination length is small ($\bar{a} < .04$). As the delamination length increases the critical load decreases and it approaches asymptotically a value, which is, approximatly about 60% of the critical load of the perfect configuration. As the delamination becomes thinner and thinner, a sharp drop in the critical load is noticed at a very small delamination length ($\bar{a} = 0.01$). This drop in the critical load continues till it reaches a constant value (20% of that of the perfect shell), at a value for the delamination length paramter of $\bar{a} = 0.1$.

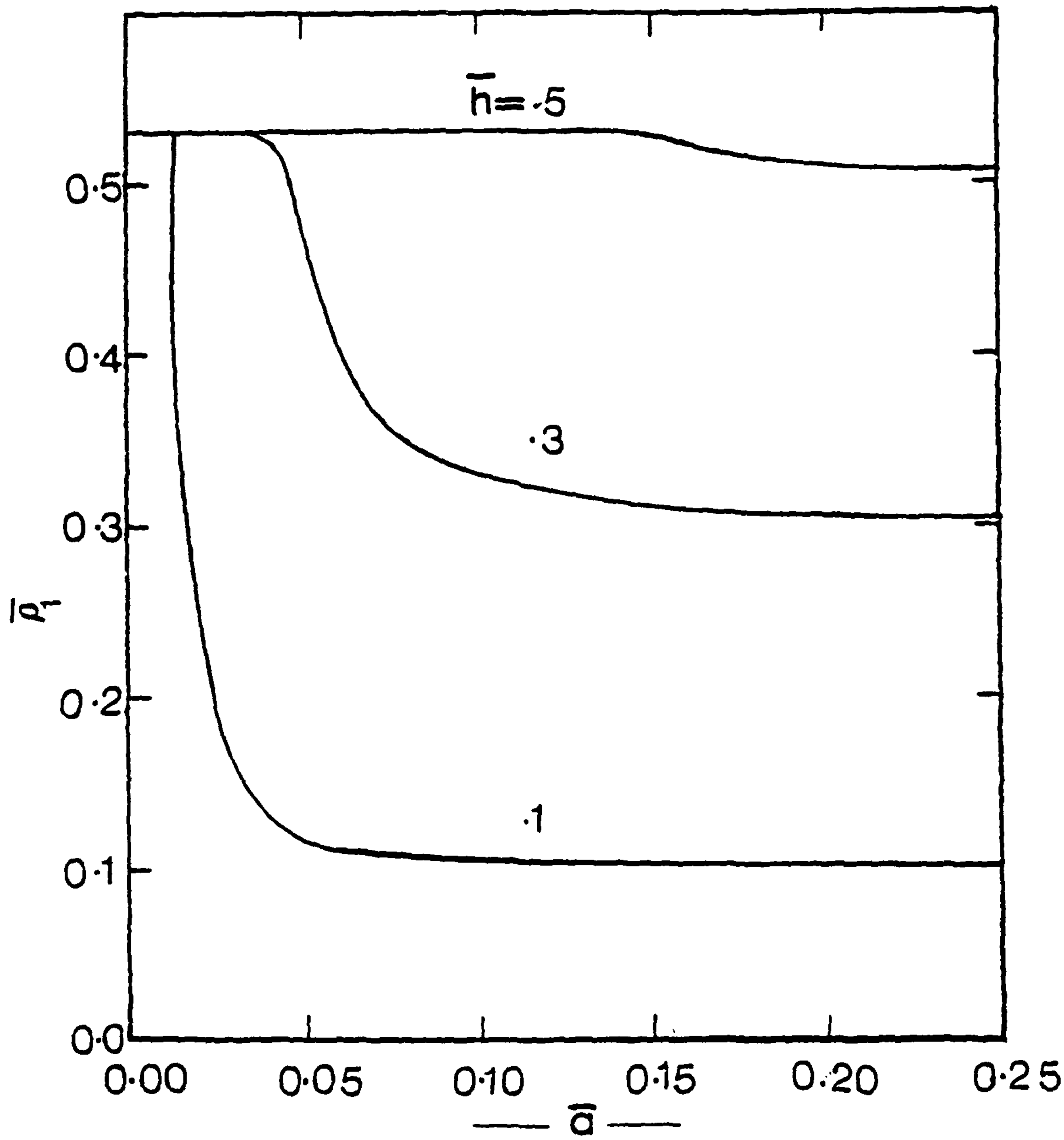


Figure 3 Effect of Delamination Length on the Critical Loads of a Cylindrical Shell with Simply Supported Ends .

The effect of lengthwise delamination position on the critical loads is also studied. Figure 4 presents results for $\bar{a} = 0.1$. For delamination thickness $\bar{h} \leqslant 0.3$, the delamination position, $\bar{\ell}$, has no appreciable effect at on the critical loads. For thicker delaminations, $\bar{h} > 0.3$, the position of delamination has an important effect on the critical load, and this effect increases as the delamination thickness increases. For instance, a reduction of about one third in the critical load is noticed for a delamination near the edge (by comparison to the critical load of a delamination far from the edge ; $\bar{\ell} > .08$), for a delamination thickness $\bar{h} = 0.5$. The position effect decreases as the delamination thickness decreases. Figure 5 shows the effect of both delamination length and position on the critical load of a delaminated cylindrical shell, with the delamination located at the mid surface of the shell ($\bar{h} = 0.5$). For a moderate delamination length, $0.1 < \bar{a} < 0.2$, it seems that the effect of delamination position is approximately the same for bothe values of delamination lengths ($\bar{a} = 0.1$ and 0.2). For shorter delamination length, the critical load decreases as the delamination moves away from the ends of the shell and it reaches a minimum at a value of $\bar{\ell}$ in the range $0.04 > \bar{\ell} > 0.02$. The exact value of $\bar{\ell}$ at which the minimum is reached depends on the delamination length parameter, $\bar{a}$. As the delamination length parameter becomes shorter and shorter, ($\bar{a} \leqslant 0.02$), the effect of delamination position can be ignored.

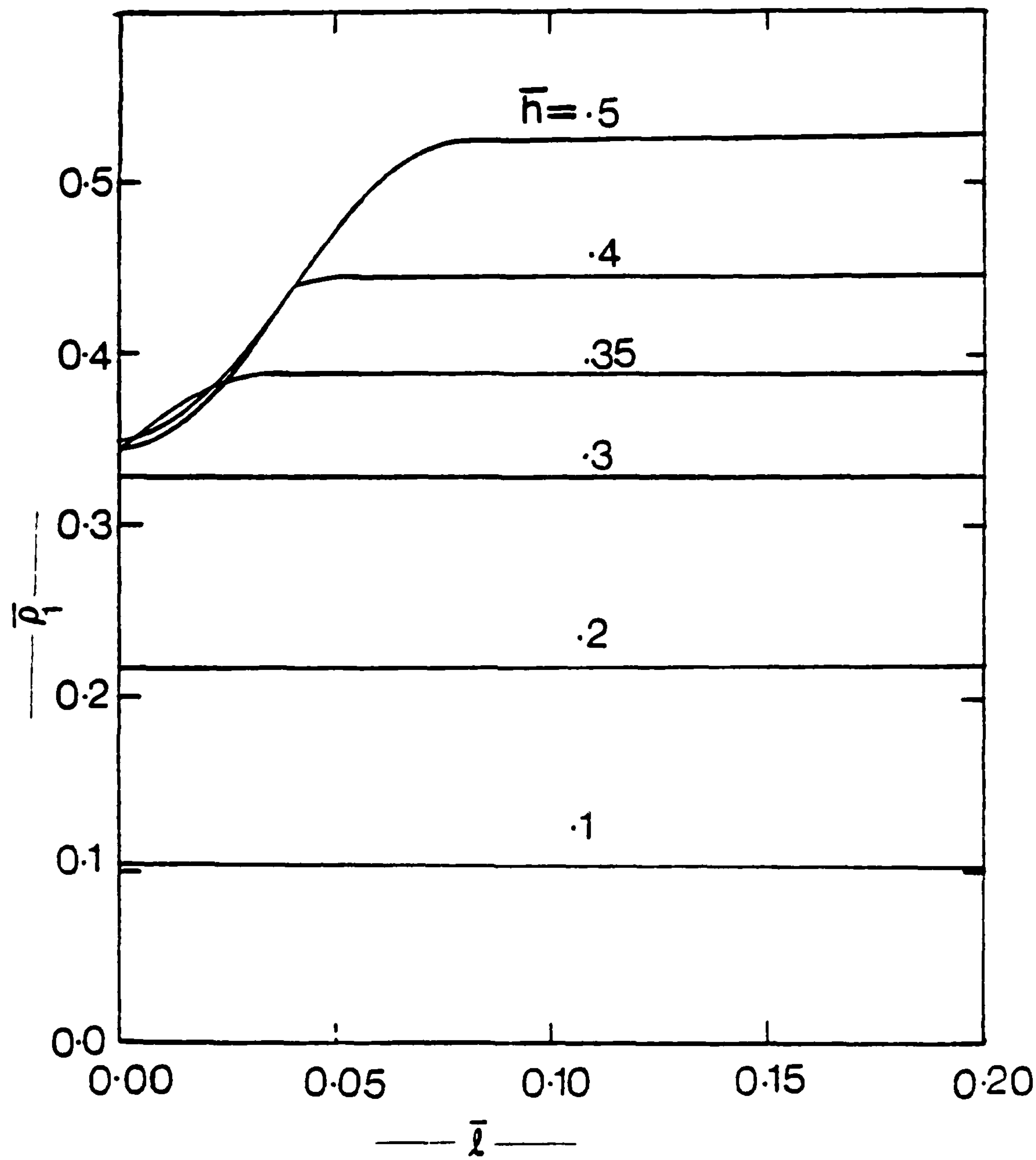


Figure 4 Effect of Delamination Position on the Critical Loads of a Cylindrical Shell with Simply Supported Ends .

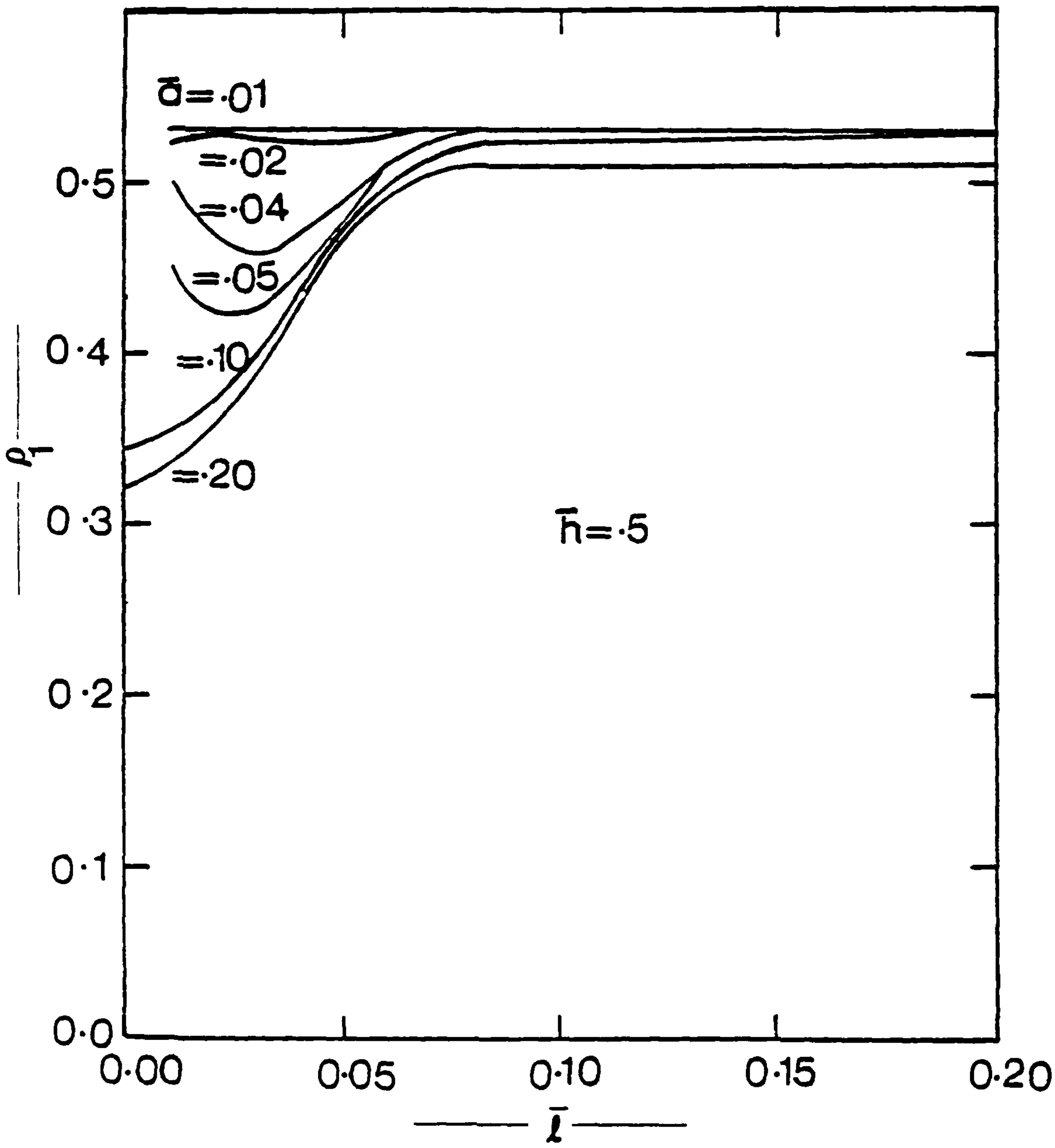


Figure 5 Effect of Delamination Position on the Critical Loads of a Cylindrical Shell with Simply Supported Ends .

A similar set of curves are obtained for clamped boundary conditions. The effect of delamination length on the critical loads of a symmetrically delaminated cylindrical shell, for different values of the delamination thickness, is shown in Figure 6. This figure, illustrates that, only for very small values for the delamination length ($\bar{a} \leqslant .01$), the delamination has no significant effect on the critical loads, regardless of the value for $\bar{h}$. For larger values for the delamination length parameter, $\bar{a}$, a noticeable drop in the critical loads is observed. For example, a drop of about one half of the critical load (of the perfect shell) is observed at a delamination thickness parameter, $\bar{h} = 0.5$. The drop of the value of the critical load is more severe for thinner delaminations. For instance, the critical load is only 10% of the critical load of the perfect shell for delamination thickness $\bar{h} = 0.1$ with delamination length $\bar{a} > .07$.

Unlike the simply supported case, the position of delamination w.r.t. the edges of the shell with clamped ends, has virtually no effect on the critical loads, for delamination length $\bar{a} = 0.1$, irrespective of its thickness, as shown on Figure 7.

The effect of delamination position on the critical loads of a delaminated cylindrical shell with the delamination located at the middle surface of the shell, for various values of delamination length is presented on Figure 8. It is observed that for moderate delamination lengths ($\bar{a} = 0.1$ and 0.2) the position of the delamination has no effect on the critical loads of the shell. On the other hand, for short delaminations ($\bar{a} = .05$), the critical load decreases as the delamination moves away from the edge of the shell.

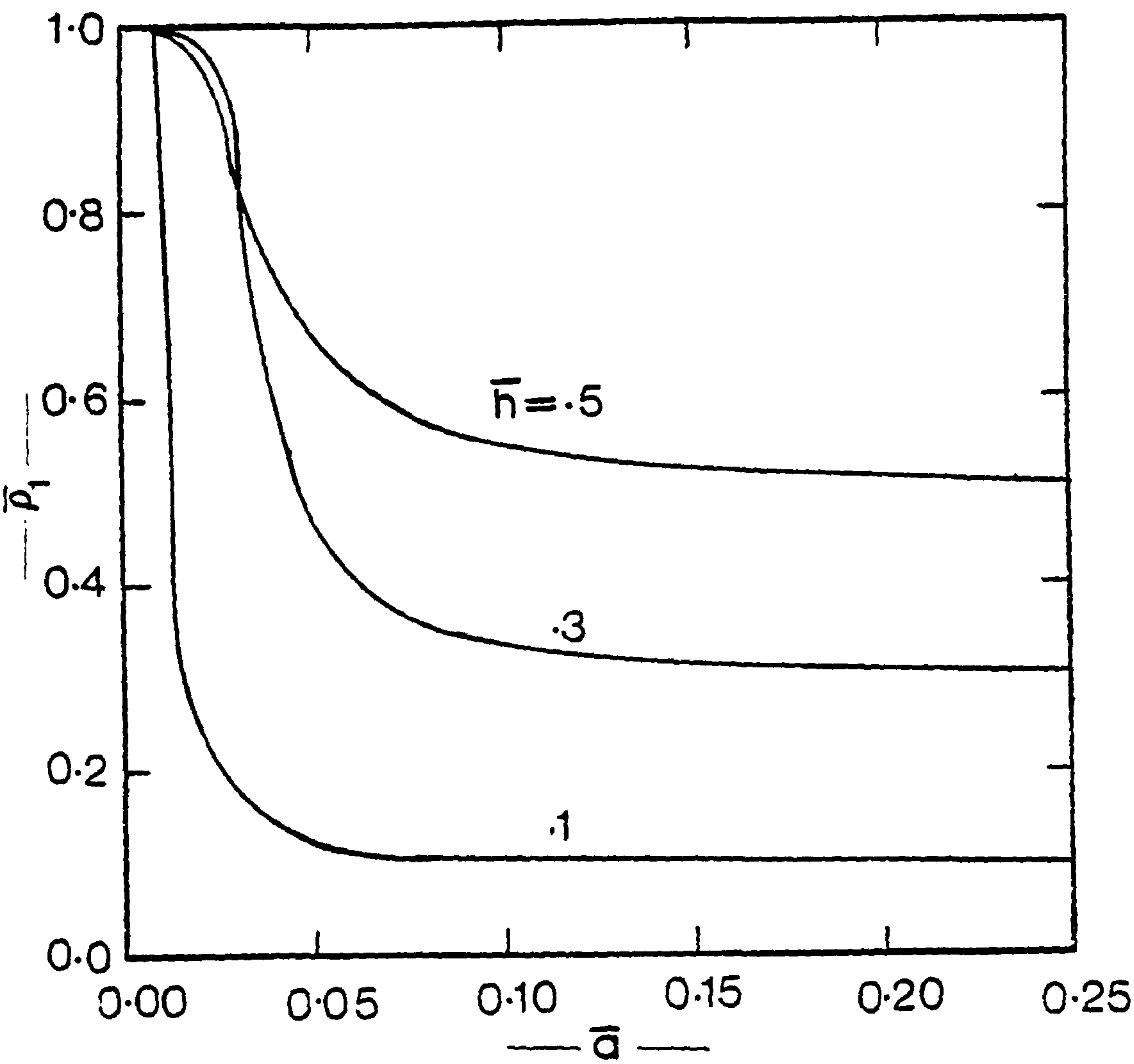


Figure 6 Effect of Delamination Length on the Critical Loads of a Cylindrical Shell with Clamped Ends .

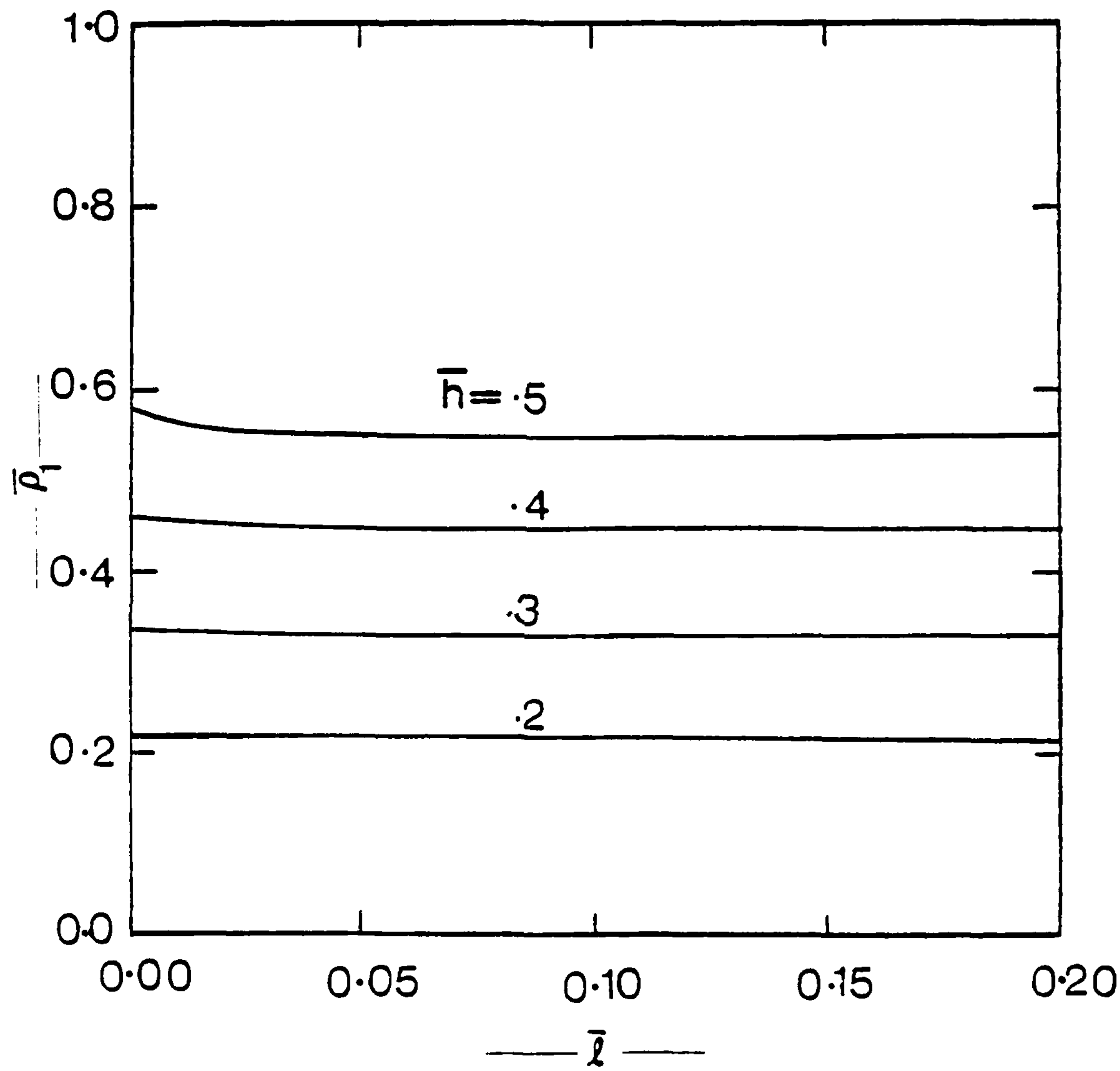


Figure 7 Effect of Delamination Position on the Critical Loads of a Cylindrical Shell with Clamped Ends .

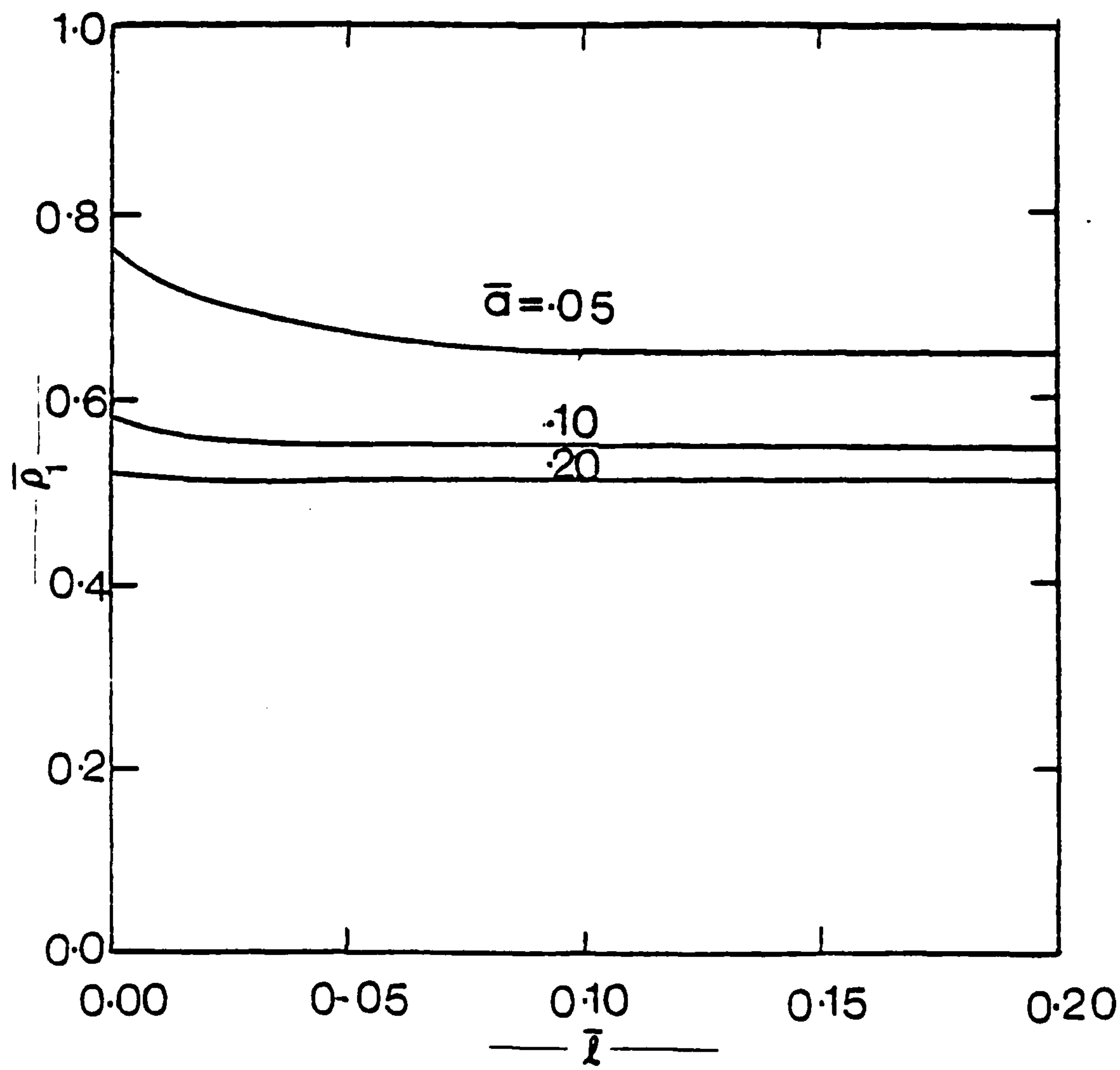


Figure 8 Effect of Delamination Position on the Critical Loads of a Cylindrical Shell with Clamped Ends .

A comparison of the results on Figures 3 and 6, shows that, for moderate and large delamination lengths ($\bar{a} > 0.1$), a delaminated cylindrical shell has the same critical load, irrespective of the boundary conditions (simply supported or clamped) and this critical load paramter is approximatly equal to $\bar{h}$. The explanation of such results is that for such delamination length paramter ($\bar{a} > 0.1$), the thin region (delaminated part) buckles while the main body of the cylindrical shell remains straight (has no appreciable bending deformation), i.e. the thin part buckles as if it had a clamped boundary condition, irrespective of the type of supports of the main cylindrical shell.

Figure 9 shows a plot of the critical load parameter for region 3 (the thin region) versus the characteristic length of region 3 ($L_3 = a / \sqrt{R\ h}$) for a very thin and long delamination ($\bar{h} = .02$, $\bar{a} = 0.6$), for clamped boundary conditions. The main purpose of presenting this curve for this configuration (very thin and very long) is to check a limitting case, which is similar to the thin film analysis of the delaminated plate. As expected the obtained curve is similar to the one obtained by Hoff [6] for the critical loads of a cylindrical shell with CC4 boundary conditions.

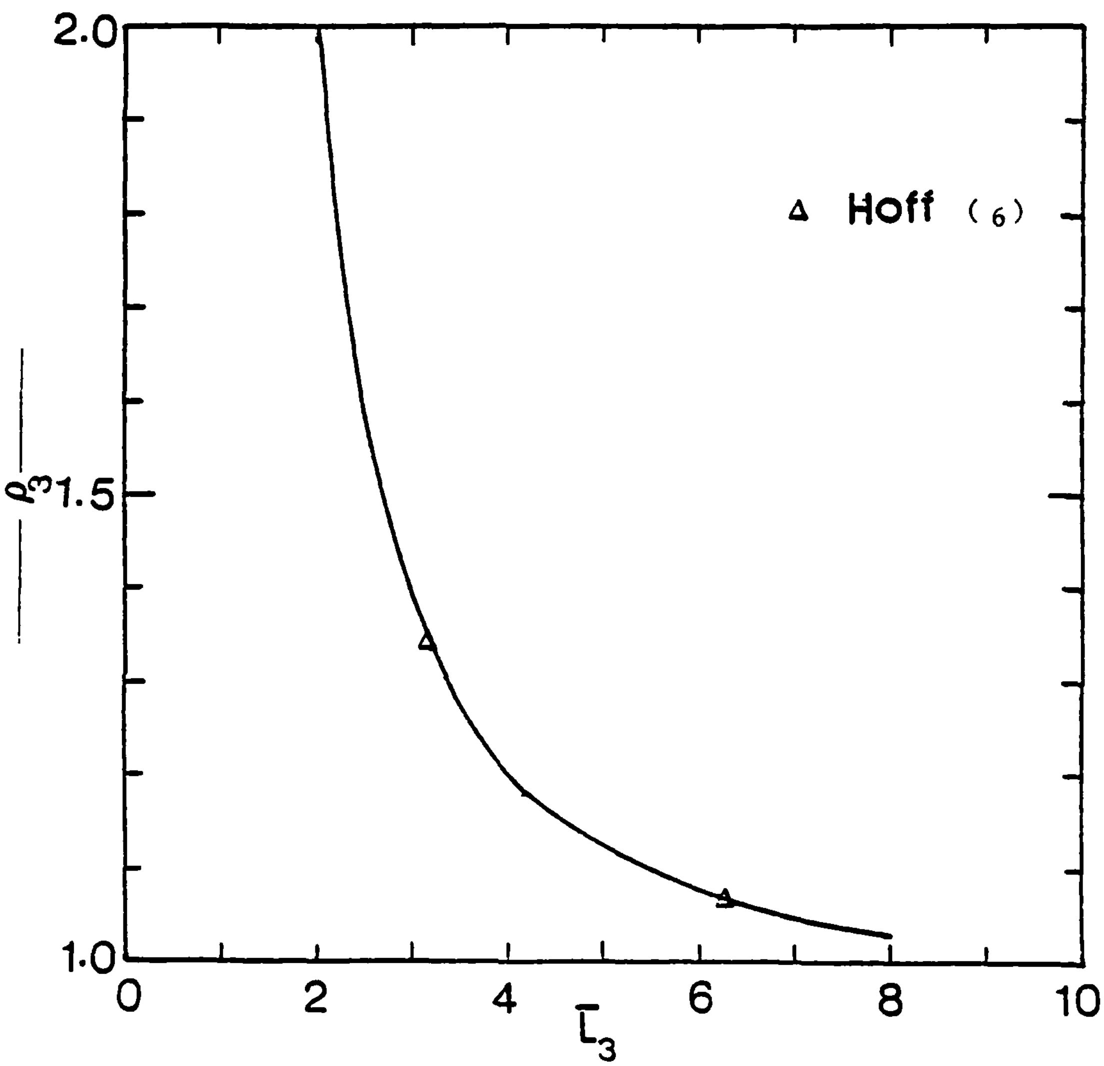


Figure 9 Critical Load Parameter for Region 3

2.6 CONCLUDING REMARKS

The investigation of the problem of buckling of delaminated cylindrical shells leads to the following observations:

1) For symmetric delamination (delamination located symmetrically w.r.t. both ends of the shell) the critical load is independent of the boundary conditions as long as the delamination length parameter, $\bar{a}$, is greater than 0.1 . For this case, $\bar{a} > 0.1$, the critical load depends only on the delamination thickness parameter, $\bar{h}$.

2) For clamped boundaries (CC-4), the position of delamination w.r.t. the ends of the cylindrical shell has no effect on the critical load for moderate and large values of the delamination length parameter, $\bar{a}>0.1$. As the delamination length becomes smaller, the effect of position of delamination becomes more pronounced, and the critical load increases as the delamination moves towards the clamped ends .

3) For simply supported boundaries (SS-1), the position of delamination w.r.t. the ends of the cylindrical shell has a significant effect on the critical load, even for small values of the delamination length parameter. Unlike the case of clamped boundaries, for simply supported boundaries (with $\bar{a} = 0.1$), the critical load decreases as the delamination moves towards the edges of the shell boundary as long as the delamination thickness, $\bar{h}$, is greater than 0.3, otherwise, the change of delamination position has a negligible effect on the critical load for smaller values of the delamination thickness, $\bar{h} < 0.3$.

APPENDIX A

CHARACTERISTIC EQUATIONS OF DELAMINATED CYLINDRICAL SHELLS

The solution to the buckling equation , Eq.(44), can be written in the form :

$$w_i = \sum_{j=1}^{8} A_{ij}\, e^{\lambda_{ij} \frac{x}{R} \left(\frac{2E_{11}}{\sigma_{c\ell}}\right)_i^{1/2}} \sin s_i \frac{y}{R}\left(\frac{2E_{11}}{\sigma_{c\ell}}\right)_i^{1/2}$$

Use of the proper boundary and continuity conditions leads to the system of linear , homogeneous algebraic equations

$$[C]\{X\} = 0$$

where $[C]$ is a 32x32 matrix and $\{X\}$ a 1x32 (column) matrix.

$$\{X\}^T = [\, A_{11}, A_{12}, A_{13}, A_{14}, A_{15}, A_{16}, A_{17}, A_{18}, A_{21}, A_{22}, A_{23}, A_{24}, A_{25}, A_{26}, A_{27}, A_{28},$$
$$A_{31}, A_{32}, A_{33}, A_{34}, A_{35}, A_{36}, A_{37}, A_{38}, A_{41}, A_{42}, A_{43}, A_{44}, A_{45}, A_{46}, A_{47}, A_{48} \}$$

The general element of the matrix $[C]$ is C_{ij} . In defining the elements C_{ij} two parameters (r,q) are used, and defined as

for $j = 1,2,3 \ldots 8$, r=1 and q=j

$j = 9,10,11,\ldots$, r=2 and q=j-9

$j = 17,18,\ldots 24$, r=3 and q=j-16

$j = 25,26,\ldots 32$, r=4 and q=j-24

Also we have,

$C_{ij} = 0$ for $i=1,2,3,\ldots$ 16 and $j = 25,26,\ldots$ 32

and for $i=17,18,19\ldots 32$ and $j = 1,2,3,\ldots$ 8

Moreover, the following parameters are defined as

$$vv = \sqrt{3(1-\nu_{12}\nu_{21})}$$

$t_1=t_4=1$; $t_2=1-\overline{h}$,; $t_3=\overline{h}$

$\delta=1$ for $r=1$ or 4

$=-1$ for $r=2$ or 3

$$f_r = D_{xx_r} / (Q_{11}\ t^3{}_r/12)$$

$r=1,2,3,4$

$$g_r = A_{xx_r} / Q_{11}\ t_r$$

$$D_j = \lambda^4_{rq} + (A_{yy}/A_{xx})_r\ S^4_{rq} + \{(A^2_{xy}+2A_{xy}A_{ss}-A_{xx}A_{yy})/A_{xx}A_{ss}\}\lambda^2_{rq}S^2_{rq}$$

$$ee_{ij} = e^{\lambda_{rq}\ell_i (L/R)(R/t)^{\frac{1}{2}}_r [12(1-\nu_{12}\nu_{21})]^{\frac{1}{4}}}$$

$i=1,2,3,\ldots$ 32

$j=1,2,3,\ldots$ 32

where $\ell_i = 0$ for $i=1,2,3,4$

$= \overline{\ell}$ for $i=5,6,7,\ldots$ 16

$= \overline{\ell}+\overline{a}$ for $i=17,18,\ldots$ 28

$= 1$ for $i=29,30,31,32$

$$uu_j = t_r^{\frac{1}{2}}\{(A_{xy}/A_{xx})_r\lambda_{rq} + (A_{yy}/A_{xx})_r\ S_{rq}\} / D_j$$

$$vv_j = t_r^{\frac{1}{2}}\{[(A^2_{xy}+a_{xy}A_{ss}-A_{xx}A_{yy})/A_{ss}A_{xx}]_r\ \lambda^2_{rq}+(A_{yy}/A_{xx})_r S^2_{rq}\}S_{rq}/D_j$$

$$Nxx_j = t_r\ \lambda^2_{rq}\ S^2_{rq} / D_j$$

$$Nxy_j = t_r\ \lambda^3_{rq}\ S_{rq} / D_j$$

$$Mxx_j = t^2_r\ f_r\ \{(D_{xy}/D_{xx})_r\ S^2_{rq} - \lambda^2_{rq}\}$$

$$Qx_j = t_r^{3/2}\ f_r\ \lambda_{rq}\ \{\lambda^2_{rq}-[(D_{xy}+2D_{ss})/D_{xx}]_r\ S^2_{rq} + 2\rho_r/f_r\}$$

The elements of the matrix [C] are obtained by multiplying each of the following elements (c_{ij}) by ee_{ij} .($C_{ij} = e_{ij} \times c_{ij}$) .

$$c_{1j} = 1 \qquad j=1,2,3,\ldots\ 8$$
$$= 0 \qquad j=9,10,11,\ldots\ 24$$

$$c^*_{2j} = \lambda_{1j} \qquad j=1,2,3,\ldots\ 8$$
$$= 0 \qquad j=9,10,11,\ldots\ 24$$

$$c^*_{3j} = uu_j \qquad j=1,2,3,\ldots\ 8$$
$$= 0 \qquad j=9,10,11,\ldots\ 24$$

$$c^*_{4j} = vv_j \qquad j=1,2,3,\ldots\ 8$$
$$= 0 \qquad j=9,10,11,\ldots\ 24$$

$$c_{5j} = \delta \qquad j=1,2,3,\ldots\ 16$$
$$= 0 \qquad j=17,18,\ldots\ 24$$

$$c_{6j} = \delta\ \lambda_{rq}/t_r^{\frac{1}{2}} \qquad j=1,2,3,\ldots\ 16$$
$$= 0 \qquad j=17,18,\ldots\ 24$$

$$c_{7j} = vv_j + h\ \nu\nu S_{rq} \qquad j=1,2,3,\ldots\ 8$$
$$= -vv_j \qquad j=9,10,11,\ldots\ 16$$
$$= 0 \qquad j=17,18,\ldots\ 24$$

$$c_{8j} = uu_j - h\ \nu\nu\ \lambda_{rq} \qquad j=1,2,3,\ldots\ 8$$
$$= -uu_j \qquad j=9,10,11,\ldots\ 16$$
$$= 0 \qquad j=17,18,19,\ldots\ 24$$

$$c_{9j} = 1 \qquad j=1,2,3,\ldots\ 8$$
$$= 0 \qquad j=9,10,11,\ldots\ 16$$
$$= -1 \qquad j=17,18,19,\ldots\ 24$$

$$c_{10j} = \lambda_{1j} \qquad j=1,2,3,\ldots\ 8$$
$$= 0 \qquad j=9,10,11,\ldots\ 16$$
$$= -\lambda_{rq}/t_r^{\frac{1}{2}} \qquad j=17,18,19,\ldots\ 24$$

$$c_{11j} = vv_j - H\,\nu\nu\ S_{rq} \qquad j=1,2,3,\ldots\ 8$$
$$= 0 \qquad j=9,10,11,\ldots\ 16$$
$$= -vv_j \qquad j=17,18,19,\ldots\ 24$$

$$c_{12j} = uu_j + H\,\nu\nu\ \lambda_{1j} \qquad j=1,2,3,\ldots\ 8$$
$$= 0 \qquad j=9,10,11,\ldots\ 16$$
$$= -uu_j \qquad j=17,18,19,\ldots\ 24$$

$$c_{13j} = \delta\ Nxx_j \qquad j=1,2,3,\ldots\ 24$$

$$c_{14j} = \delta\ Nxy_j \qquad j=1,2,3,\ldots\ 24$$

$$c_{15j} = Mxx_j \qquad j=1,2,3,\ldots\ 8$$
$$= -Mxx_j - h(\nu\nu/1-\nu_{12}\nu_{21})\ g_2\ Nxx_j$$
$$j=9,10,11,\ldots\ 16$$
$$= -Mxx_j + H(\nu\nu/1-\nu_{12}\nu_{21})\ g_3\ Nxx_j$$
$$j=17,18,19,\ldots\ 24$$

$$c_{16j} = \delta\ Qx_j \qquad j=1,2,3,\ldots\ 24$$

$$
\begin{aligned}
c_{17j} &= 0 && j=9,10,11,\ldots\ 24 \\
&= 1 && j=25,26,27,\ldots\ 32 \\[1ex]
c^*_{18j} &= 0 && j=9,10,11,\ldots\ 24 \\
&= \lambda_{4j} && j=25,26,27,\ldots\ 32 \\[1ex]
c^*_{19j} &= 0 && j=9,10,11,\ldots\ 24 \\
&= uu_j && j=25,26,27,\ldots\ 32 \\[1ex]
c^*_{20j} &= 0 && j=9,10,11,\ldots\ 24 \\
&= vv_j && j=25,26,27,\ldots\ 32 \\[1ex]
c_{21j} &= -1 && j=9,10,11,\ldots\ 16 \\
&= 0 && j=17,18,19,\ldots\ 24 \\
&= 1 && j=25,26,27,\ldots\ 32 \\[1ex]
c_{22j} &= -\lambda_{rq}/\ t_r^{1/2} && j=9,10,11,\ldots\ 16 \\
&= 0 && j=17,18,19,\ldots\ 24 \\
&= \lambda_{4j} && j=25,26,27,\ldots\ 32 \\[1ex]
c_{23j} &= -\ vv_j && j=9,10,11,\ldots\ 16 \\
&= 0 && j=17,18,19,\ldots\ 24 \\
&= vv_j + h\ \nu\nu\ S_{rq} && j=25,26,27,\ldots\ 32 \\[1ex]
c_{24j} &= -\ uu_j && j=9,10,11,\ldots\ 16 \\
&= 0 && j=17,18,19,\ldots\ 24 \\
&= uu_j - h\ \nu\nu\ \lambda_{rq} && j=25,26,27,\ldots\ 32
\end{aligned}
$$

$c_{25j} = 0$ $\qquad j=9,10,11,..\ 16$

$= \delta$ $\qquad j=17,18,19,..\ 32$

$c_{26j} = 0$ $\qquad j=9,10,11,..\ 16$

$= \delta\ \lambda_{rq}/\ t_r^{1/2}$ $\qquad j=17,18,19,..\ 32$

$c_{27j} = 0$ $\qquad j=9,10,11,..\ 16$

$= -\ vv_j$ $\qquad j=17,18,19,..\ 24$

$= vv_j\ -\ H\ \nu\nu\ S_{rq}$ $\qquad j=25,26,27,..\ 32$

$c_{28j} = 0$ $\qquad j=9,10,11,..\ 16$

$= -\ uu_j$ $\qquad j=17,18,19,..\ 24$

$= uu_j\ +\ H\ \nu\nu\ \lambda_{rq}$ $\qquad j=25,26,27,..\ 32$

$c_{29j} = \delta\ Nxx_j$ $\qquad j=9,10,11,..\ 32$

$c_{30j} = \delta\ Nxy_j$ $\qquad j=9,10,11,..\ 32$

$c_{31j} = -\ Mxx_j\ -\ h\ (\nu\nu/1-\nu_{12}\nu_{21})\ g_2\ Nxx_j$

$j=9,10,11,..\ 16$

$= -\ Mxx_j\ +\ H\ (\nu\nu/1-\nu_{12}\nu_{21})\ g_3\ Nxx_j$

$j=17,18,19,..\ 24$

$= Mxx_j$ $\qquad j=25,26,27,..\ 32$

$c_{32j} = \delta\ Qx_j$ $\qquad j=9,10,11,..\ 32$

The above elements (c_{ij}) corespond to the clamped supports case of CC-4 . For simply supported boundaries of SS-1 the elements with the super star "*" must change to ,

$$c_{2j} = \lambda_{1j}^2 \qquad j=1,2,3,..\quad 8$$

$$= 0 \qquad j=9,10,11,..\quad 24$$

$$c_{3j} = Nxx_j \qquad j=1,2,3,..\quad 8$$

$$= 0 \qquad j=9,10,11,..\quad 24$$

$$c_{4j} = Nxy_j \qquad j=1,2,3,..\quad 8$$

$$= 0 \qquad j=9,10,11,..\quad 24$$

$$c_{18j} = 0 \qquad j=9,10,11,..\quad 24$$

$$= \lambda_{4j} \qquad j=25,26,27,..\quad 32$$

$$c_{19j} = 0 \qquad j=9,10,11,..\quad 24$$

$$= Nxx_j \qquad j=25,26,27,..\quad 32$$

$$c_{20j} = 0 \qquad j=9,10,11,..\quad 24$$

$$= Nxy_j \qquad j=25,26,27,..\quad 32$$

APPENDIX B

COLUMNS PARTIALLY SUPPORTED

BY ELASTIC FOUNDATION

Structural elements supported elastically along part or along their entire lengths are used very offen in structure confinurations. The elastic supports may be a foundation as in railroad tracks, or adjacent elastic structural elements such as in stiffened plates and shells [7]. Moreover, elastic supports can also be found in delaminated structures when the delaminated region (or part of it) deforms towards the main body of the shell.

Although there are many models to represent the elastic support (which will be referred to as the elastic foundation from here on), [8-13], the simplest of all these models is known as the Winkler foundation [8].

The considered elastic foundation is assumed to be of the Winkler type [8], where the foundation is approximated by a series of closely packed linear springs, and the foundation reaction at any point is proportional to the deflection at that point.

A column of total length L, has parts of length ℓ, which are surrounded by the elastic foundatiopn, so that the elastic foundation has the same lateral deflection as the column, for these parts. The rest of the column is not supported elastically (see Figure B-1).

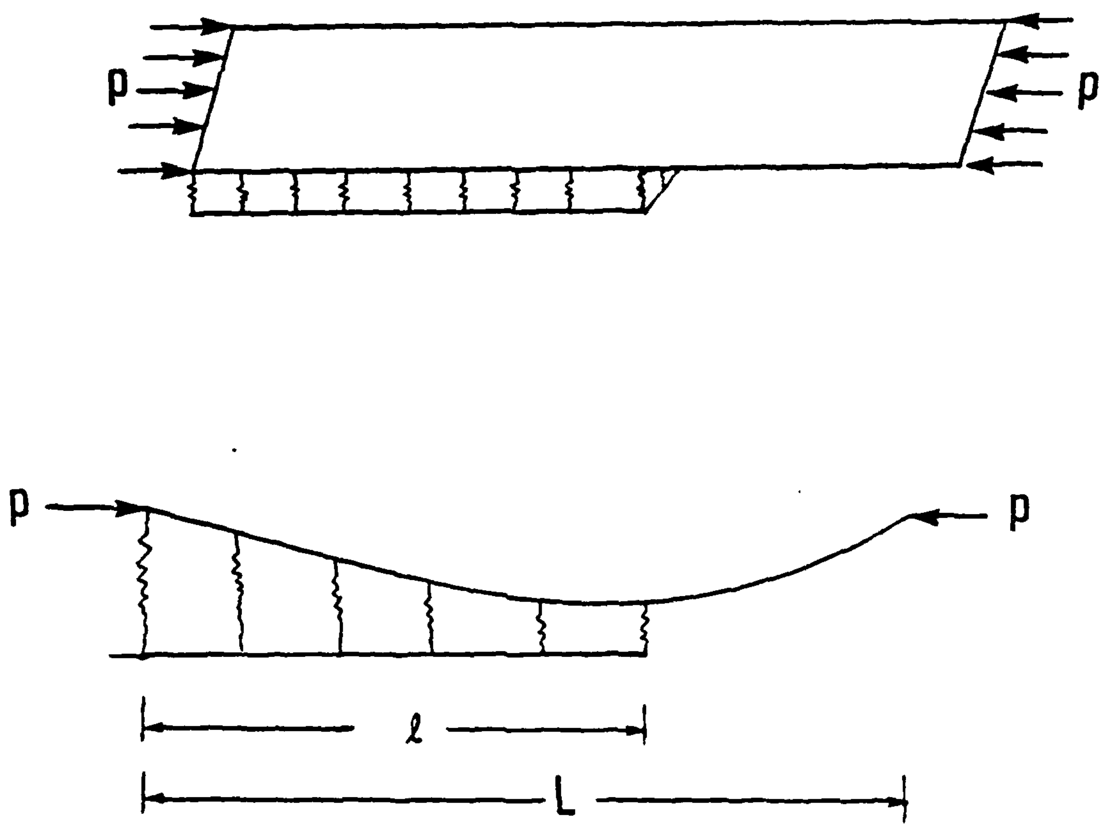


Figure B-1 Column Partially Suppored by Elastic Foundation

The buckling equations that govern the behavior of the two regions (with and without the elastic foundation) of the axially column are :

$$w_{,xxxx} + \frac{p}{EI} w_{,xx} + \frac{\beta}{EI} w = 0 \qquad (B\text{-}1)$$

$$0 < x < \ell$$

$$w_{,xxxx} + \frac{p}{EI} w_{,xx} = 0 \qquad (B\text{-}2)$$

$$\ell < x < L$$

wher β is the modulus of elastic foundation

The solution of the buckling equation, Eq.(B-1), takes one of the following three forms ,depending on the value of the load parameter ($\gamma = p/2\sqrt{\beta EI}$) , (for details see Ref.39)

Case I :

$$\gamma > 1$$

$$w_1 = b_1 \cos k_1 x + b_2 \sin k_1 x + b_3 \cos k_2 x + b_4 \sin k_2 x \qquad (B\text{-}3i)$$

where

$$k_{\substack{1\\2}} = (\beta/EI)^{\frac{1}{2}} (\gamma \mp \sqrt{\gamma^2 - 1})^{\frac{1}{2}}$$

Case II :

$$\gamma = 1$$

$$w_1 = b_1 \cos k_3 x + b_2 \sin k_3 x + b_3\, x \cos k_3 x + b_4\, x \sin k_3 x$$

(B-3ii)

where

$$k_3 = (\beta/EI)^{\frac{1}{2}}$$

Case III :

$$\gamma < 1$$

$$w_1 = b_1\, e^{(s+it)x} + b_2\, e^{-(s+it)x} + b_3\, e^{(-s+it)x} + b_4\, e^{(s-it)x}$$

where (B-3iii)

$$\substack{s\\t} = (\beta/EI)^{\frac{1}{2}} \sqrt{(1 \mp \gamma)/2}$$

The solution of the buckling equation, Eq.(B-2), is of the form

$$w_2 = a_1 \cos kx + a_2 \sin kx + a_3 x + a_4 \qquad (B\text{-}4)$$

$$L > x > \ell$$

where

$$k = \sqrt{p/EI}$$

The related boundary and continuity conditions are

<u>Boundary conditions</u>

Either	Or
$w = 0$	$w_{,xxx} + \frac{p}{EI} w_{,x} = 0$
$w_{,x} = 0$	$w_{,xx} = 0$

(B-5)

<u>Continuity Conditions</u> (at $x=\ell$)

$$w_1 = w_2$$

$$w_{1,x} = w_{2,x}$$

$$w_{1,xx} = w_{2,xx} \qquad (B\text{-}6)$$

$$w_{1,xxx} = w_{2,xxx}$$

The solution of the buckling equations, Eqs.(B-3) and (B-4) requires knowledge of eight constants (b_i, a_i, $i=1,2,3,4$). There exist eight boundary and continuity conditions. The use of the boundary and continuity conditions yields a system of linear, homogeneous, algebraic equations in b_i and a_i. For a nontrivial solution to exist the determinant of the coefficients must vanish.

Results are obtained for axially compressed columns with simply supported boundary conditions. The same solution scheme is also used to obtain critical loads for a simply supported column with one (two) third of its total length elastically supported. Moreover, results are presented for columns divided into four equal parts, two of them being elastically supported.

The obtained results ar› presented graphically on Figures(B-2) and (B-3). On Figure (B-2) the critical load, normalized w.r.t. p_E ($p_E = \pi^2 EI/L^2$), for a column with part of length ℓ, attached to an elastic foundation, while the rest of the column is not. As the length of the portion of the column which is attached to the elastic foundation increases, the critical load increases. Of course all the curves lie between the two limiting cases, the straight line, $\bar{P} = 1.0$, which corresponds to the case where there is no elastic foundation at all, and the curve presented by the broken line ,which corresponds to the case where the column is fully supported by the elastic foundation.

Figure(B-3) presents the critical load parameter, P_{cr}, versus the normalized modulus of foundation, $\bar{\beta}$. On Figure(B-3), the column is assumed to be devided into equal parts (two,three and four) and one (two) part(s) of the column is elastically supported. These curves are obtained primarily to have an (approximate) idea about what changes are expected to happen to the critical load of the delaminated cylindrical shell, if there is a contact between the two regions across the delamination. As the results of Figure(B-3) show, the value of the critical load depends on the wave number, the dimension of the structural element, as well as on the (equivalent) modulus of foundation

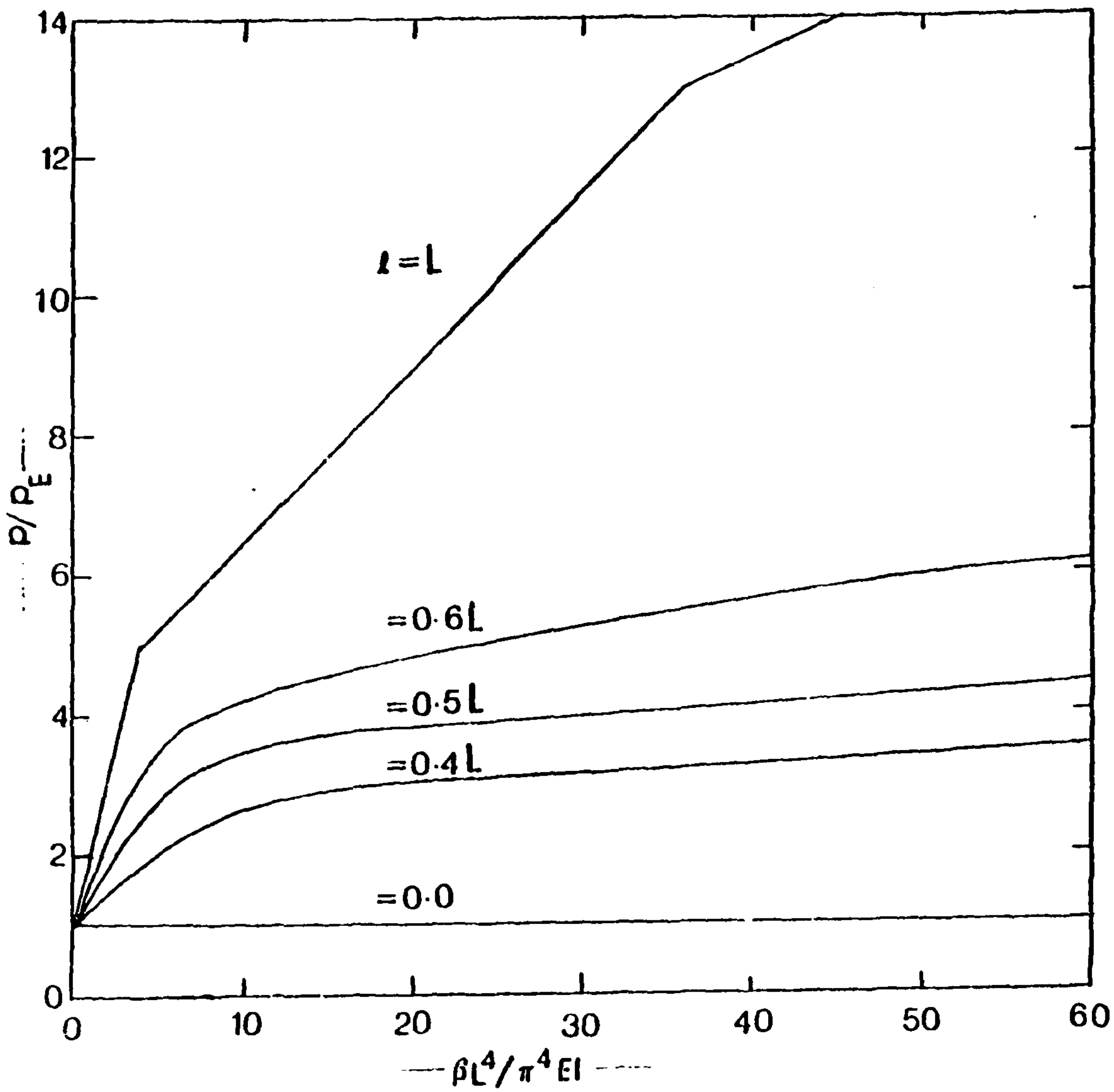


Figure B-2. Column Partially Supported by Elastic Foundation

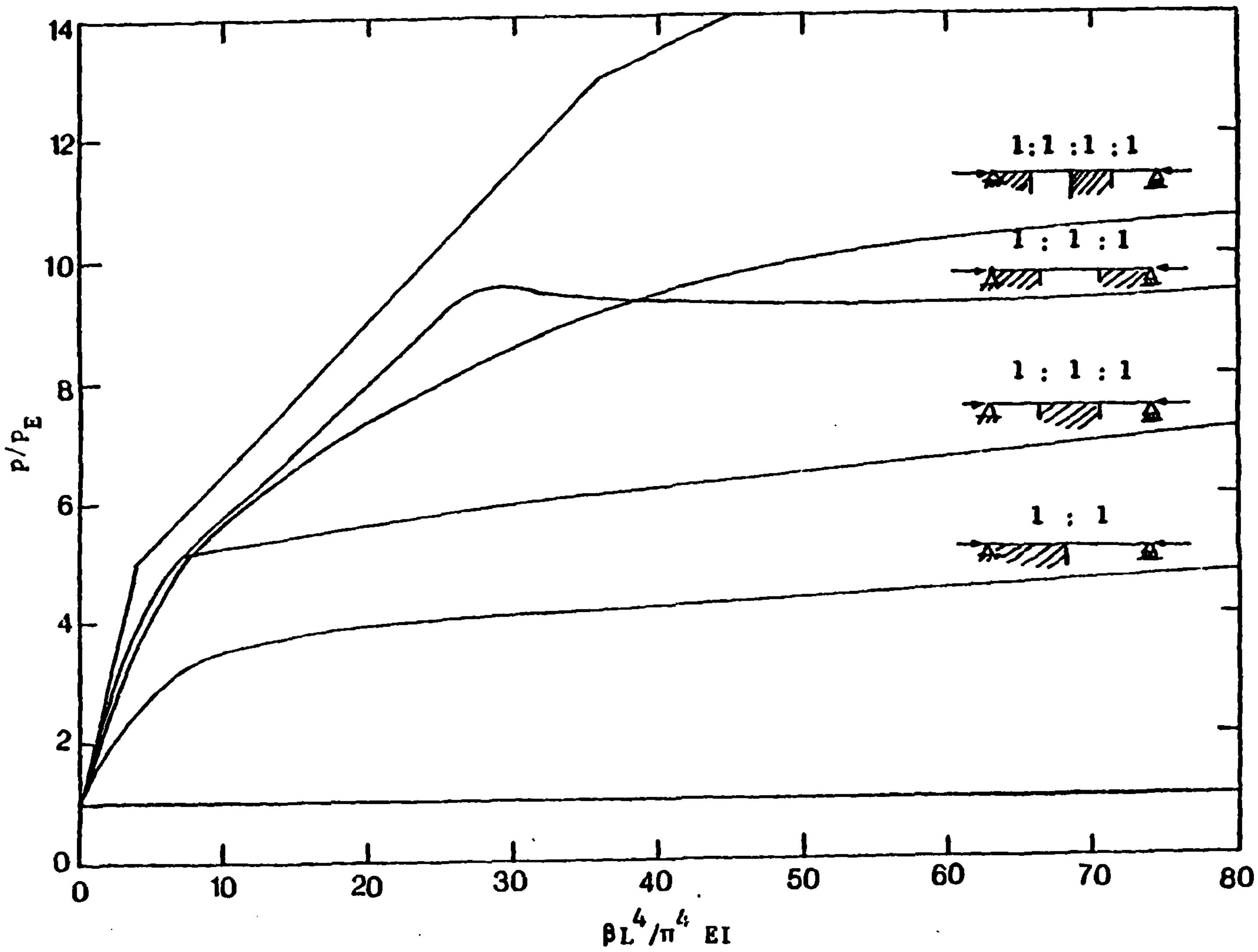


Figure B-3. Column Partially Supported by Elastic Foundation

of the element. All the parameters, that affect the critical load, can be found for a specific material, except the modulus of foundation which has to be determined experimentally.

REFERENCES

1. Simitses,G.J. and Sallam,S., "Delamination Buckling and Growth of Flat Composite Structural Elements," AFOSR,TR-84XXX Georgia Institute of Technology, Atlanta, Georgia, 1984.

2. Kulkarni,S.V. and Frederick,D., "Buckling of Partially Debonded Layered Cylindrical Shell," Proceedings of AIAA/ASME/SAE 14th SDM Conference, Williamsburg, Virginia, March 1973.

3. Jones, R. M., "buckling of Stiffened Two-Layered Shells of Revolution with a Circumferentially Cracked Unbonded Layer," AIAA J. Vol. 7, 1969, pp. 1511-1517.

4. Troshin,V.P., "Effect of Longitudinal Delamination in a Laminar Cylindrical Shell on the Critical External Pressure," J. of Comp. Materials, Vol. 17, No. 5, 1983, pp. 563-567.

5. Calcote, L. R., The Analysis of Laminated Composite Structues, Van Nostrand Reinhold Company, New York, 1969 (Chs.2&6) .

6. Hoff, N. J. and Soong T. C., "Buckling of Circular Cylindrical Shells in Axial Compression," Int. J. Mech. Sci. , Vol. 7, 1965, pp. 489-520.

7. Simitses,G.J., Elastic Stability of Structures, Pretice-Hall Inc. , Englewood , Cliffs,N.J., 1976 (Ch.6).

8. Hetenyi, M., Beams on Elastic Foundation, The University of Michigan Press, Ann Arbor, 1946.

9. Biot, M.A., "Bending of Infinite Beam on Elastic Foundation," J. Appl. Mech., Vol. 59, 1937, pp. A1-A7.

10. Kerr, A.D., "elastic and Viscoelastic Models," J. Appl. Mech., Sept., 1964, pp. 491-498.

11. Reissner, E., "A Note on Deflection of Plates on a Viscoelastic Foundation," J. Appl. Mech., Vol. 25, Transaction ASME, March, 1958, pp. 144-145.

12. Fletcher, D.Q. and Hermann, L.R., "Elastic Foundation Representation of Continuum," J. Eng. Mech. division, Proceedings ASCE, Feb., 1971, pp. 95-107.

13. Seide, P., "Compressive Buckling of a Long Simply Supported Plate on an Elastic Foundation," J. Aero. Sci., Vol. 25, June 1958, p. 382.

Part B: "Buckling of Multi-Annular Plate"

1. INTRODUCTION

The stability of multi-annular plates is being considered. Most researchers investigated the buckling of a single annular plate (with or without a rigid inclusion) when subjected to, primarily, a uniform stress field.

The elastic stability of an annular plate subjected to shear forces, distributed along the edges, was considered by Dean [1]. Stability due to bending moment caused by initial stress was considered by Willers [2]. Federhofer [3,4], Olsson [5], and Eggar [6] dealt with the buckling of annular plates of varying thickness. Stability analyses under a uniform compressive load were considered by Olsson [7], Schubert [8] for clamped and simple-supported plates, respectively, and by Yamaki [9] for various boundary conditions. Yamaki showed that under some B.C.'s the axisymmetric mode does not yield the lowest critical load, and therefore higher modes must be investigated.

Meissner [10] dealt with axisymmetric buckling of an annular plate, simply-supported or clamped at the outer boundary and free at the inner one. Most of the above references used Bessel functions to solve the buckling equation. Mansfield [11] dealt with an infinite circular plate with a hole supported along two concentric circles and subjected to radial inplane loading along the inner circles. The solution achieved was of the closed form. The same type of problem was considered by Bareeva [12-14] and by Lizarev [15-17]. Bareeva [12-14] analyzed a single annular plate with a very rigid ring at the outer boundary. A power series solution was used. The results are given for the first and second modes only. Lizarev [16-17] employed the hypergeometric function for formulating the general solution. Results are shown only for the first and second modes. A comparison of

theoretical and test results was presented by Majumdar [18] for a single annular plate. The solution for the first two modes was achieved by using Bessel functions. For the other modes, a Galerkin procedure was employed. The convergence of the Galerkin solution is limited to $0.2 < r_{inner} / r_{outside} < 0.6$.

Tie results appearing in the above references are usually limited to the first two modes and are only applicable to a single annular plate. The results are so limited because of difficulties in the mathematical formulation and the employed solution schemes.

The present investigation deals with the buckling problem of a multi-annular circular plate with different material properties and thicknesses. The in-plane loading is axisymmetric and many combinations of transverse and inplane boundary conditions are considered.

The stability analysis employs a power series type of solution to describe the buckling modes (functions). The power series method is used because of its applicability to the various B.C.'s and different properties and because it enables a general formulation for the multi-annular problem (buckling analysis).

The described formulation and solution procedure are general. Results are presented only for solid and annular circular plates. The manuscript outlines the basic assumptions, the solution scheme, and the principal parameters affecting both the critical load and the type of solution. The complete mathematical formulation is decribed, with detail.

The mathematical formulation is based on the following simplifying assumptions:

(i) The material is linearly elastic, satisfying Hooke's Law.

(ii) The properties are constant within each annular section.

(iii) The stresses are smaller than the proportional limit.

(iv) The usual Kirchhoff-Love hypotheses are applicable.

(v) Each annular section of the plate is thin (the ratio of thickness to outer radius is small by comparison to one).

(vi) The deformation gradients and the rotations are small.

2. ANALYSIS OF A MULTI-ANNULAR PLATE

The buckling analysis of multi annular, thin plate, subjected to axisymmetric loading, is described by the solution of the following equations for each annular part (see Appendix A.3).

$$(rN^{o}_{rr})_{,r} - N^{o}_{\theta\theta} = 0 \qquad (1)$$

$$N^{o}_{\theta\theta,\theta} = 0$$

$$D\,\nabla^{4}w = N^{o}_{rr}\,w_{,rr} + N^{o}_{\theta\theta}\left(\frac{w_{,\theta\theta}}{r^{2}} + \frac{w_{,r}}{r}\right) \qquad (2)$$

where N^{o}_{rr}, $N^{o}_{\theta\theta}$ are primary state stress resultants in the radial and circumferential directions, respectively; w is the transverse displacement; $(\ )_{,r}$, $(\ )_{,\theta}$ are partial derivatives with respect to radial or angular coordinates; and D is the flexural rigidity.

Eqs. (1) govern the primary state. The other equation, Eq. (2), is the buckling equation and is derived through linearization of the transverse equilibrium equation. The primary state is first solved for the entire plate. Then, use of these solutions (N^{o}_{rr}, $N^{o}_{\theta\theta}$), in Eq. (2), leads to the buckling analysis.

2.1 Primary State

The governing equilibrium equations of the primary state, for each annular section, are given by Eqs. (1). The in-plane stress distribution (see Fig. 1) is determined through in-plane equilibrium, Eq. (1), and the following in-plane displacement and force continuity conditions of the adjoining boundaries (of neighbouring annular plates):

$$u_{ii} = u_{i+1i}$$
$$N^{o}_{rr_{ii}} = N^{o}_{rr_{i+1i}} \qquad (3)$$

where:

u_{ii}, u_{ii+1} are the radial displacements of plates i and i+1 at the common joint, i.

$N_{rr_{ii}}^{o}$, $N_{rr_{i+1i}}^{o}$ are the radial stress resultants at joint i of plate i and i +1.

The stress distribution is derived by using the stiffness approach (for details see Ref. 20). The equations, using matrix notation, are:

$$\underset{\sim}{S}\,\underline{u} = \underline{p}_e \qquad (4)$$

where: $\underset{\sim}{S}$ is the stiffness matrix $[s_{\ell j}]$ $\ell, j = 0, 1 \ldots, N-1$;

u is the displacement vector at the various joint, $[u_\ell]$;

$\underline{p}_e$ is the boundary stress resultant vector at various bounding circles $[p_{e_\ell}]$; and ℓ, j are integer subscripts

The sign convention for displacements and forces is described on Fig. 1. The stiffness matrix is not symmetric and the relation between the off-diagonal terms is:

$$r_\ell s_{\ell j} = r_j s_{j\ell} \tag{5}$$

where r_ℓ r_j are the radii at joints ℓ and j. The stress field, for an annular plate i, is given by (see Ref. 20)

$$N^o_{rr_i} = -N_{o_i} + N_{oc_i}/r^2$$

$$N^o_{\theta o_i} = -N_{o_i} - N_{oc_i}/r^2$$

$$N^o_{r\theta_i} = 0 \tag{6}$$

$$N_{o_i} = (N_{rr_{ii-1}} \ \beta_i^2 - N_{rr_{ii}})/\ (\beta_i^2-1)$$

$$N_{oc_i} = \beta_i^2 r_i^2\ (N_{rr_{ii-1}} - N_{rr_{ii}}\ )/(\beta_i^2 - 1)$$

where: $N^o_{rr_i}$, $N^o_{\theta\theta_i}$, $N^o_{r\theta_i}$ are the radial, circumferential and shear stresses in plate i.

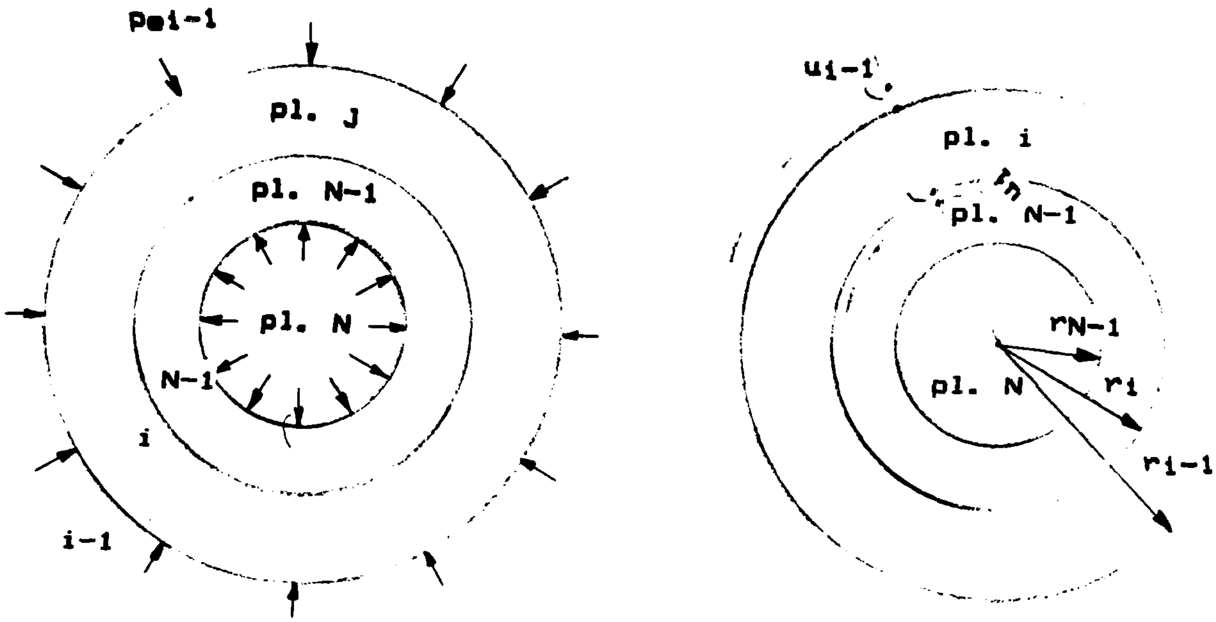
$\beta_i = r_{i-i}/r_i$ - ratio of outer to inner radii for plate i.

r_{i-1}, r_i - outer and inner radii of plate i.

The stress field in the central part is uniform and equals:

$$N^o_{rr_N} = N^o_{\theta\theta_N} = - N^o_{rr_{N-1\ N-1}} \tag{7}$$

where $N^{o}_{rr_{N-1\ N-1}}$ is positive if acting outward.



a) Forces

b) Deformations and Geomtry

Fig. 1 Forces and Deformation in Annular Plate

2.2 Buckling State

The buckling equation for each annular or circular part is given by Eq. (2). The detailed solution of the equation using a power series procedure is described in Chapter 3. The characteristic equation is obtained through the satisfaction of the boundary conditions, continuity at common joints and kinematic constraints, if they exist (see Fig. 2).

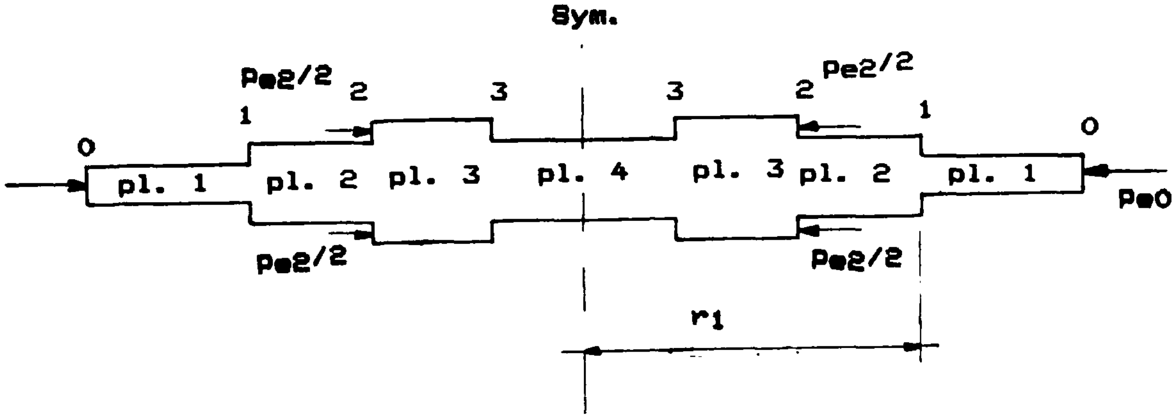


Fig. 2 Loading and Section of the plate

The continuity conditions which must be satisfied at joint i, $r = r_i$ are:

$$
\begin{aligned}
w_{ii} &= w_{i+1i} \\
w_{ii,r} &= w_{i+1i,r} \\
M_{ii} &= M_{ii+1} \\
Q_{ii} &= -Q_{i\ i+1}
\end{aligned}
\tag{8}
$$

where (see Ref. 19)

$$M_{ii} = [w,_{rr} + \frac{\nu_i}{r} (w,_r + w_{\theta\theta}/r)]\ D_i \qquad \text{at } r = r_i \tag{9}$$

$$Q_{ii} = \{ (\nabla^2 w),_r + \frac{1 - \nu_i}{r} [(\frac{1}{r} w,_\theta),_\theta] \} D_i + \frac{N^O_{rr_{ii}}}{D_i} w_{i,_r} \text{ at } r = r_i \tag{10}$$

and $D_i = E_i t_i^3/12\ (1 - \nu_i^2)$;

$(\ )_{i+1\ i}$ is a "plate i+1" parameter evaluated at joint i.

(The first index designates the plate and the second the joint);

M_{rr}, Q_r are the bending moment and shear force resultants respectively, and $N^o_{rr_{ii}}$ is the radial stress resultant of plate i at joint i.

The conditions of the boundaries are as follows:

Edge boundaries:

Clamped: $w = w_{,r} = 0$

Simply-supported: $w = M = 0$ [see Eq. (8))]

Free: $M = Q = 0$ [See Eqs. (8) - (10)]

Free to deflect not to rotate: $w_{,r} = Q = 0$ [see Eqs. (8) and (10)].

Free to rotate but elastically supported in transverse direction:

$M = Q - k_w w = 0$ (k_w is the extensional spring constant in the transverse direction).

Elastically restrained against rotation and transverse translation:

$M - k_{rot} w_{,r} = Q - k_w w = 0$

(k_{rot} is the rotational spring constant).

If any kinematic constraints exist at an interior joint, i, the following equations replace Eqs. (8).

Transverse (knife edge) support: $w_{ii} = w_{i+1i} = 0$, $w_{ii,r} = w_{i+1i,r}$;

$$M_{ii} = M_{ii+1}$$

Completely fixed : $w_{ii} = w_{i+1i} = w_{ii,r} = w_{i+1i,r} = 0$

Transverse extensional elastic support: $w_{ii,r} = w_{i+1i,r}$, $M_{ii} = M_{i+1i}$, $Q_{ii} - Q_{i+1i} - k_{w_i} w_{ii} = 0$ (k_{w_i} is the extensional spring constant at joint i in the transverse direction).

The deflection of each annular or circular part, w_i, is described by:

$$w_i(r,\theta) = \sum_{n=0}^{\infty} [\, B_{1ni}F_{1ni}(r) + B_{2ni}F_{2ni}(r) + B_{3ni}F_{3ni}(r) + B_{4ni}F_{4ni}(r)] \cos n\theta \qquad (11)$$

where:

$F_{kni}(r)$ denote the four independent solutions for the plate, corresponding to the wave numbers, n; k assumes the values of 1,2,3 and 4 to the four independent solutions. B_{kni} denote constants to be determined through boundary and joint conditions.

The characteristic equation is solved by using the Newton-Raphson procedure. In cases where the convergence is poor a Regula-Falsi or bi-section method is used (See [21]).

The details and a clarification for the employed symbols is presented in the subsequent sections.

3. SOLUTION OF THE BUCKLING EQUATIONS

A power series solution is used to solve the fourth order ordinary differential equation, derived after separation of variables. This type of solution is chosen, because of the versatility of the solution, and its applicability to the various boundary and joint conditions, and the various

types of loading. The separation of variables is achieved by assuming the deflection to be of the form,

$$w_i(r,\Theta) = \sum_{n=0}^{\infty} F_{ni}(r) \cos n\Theta \tag{12}$$

where r, θ are the radial and angular coordinates, and $F_{ni}(r)$ are the deflection functions in the radial direction (corresponding to plate i and wave number n).

Substitution of Eq. (6) into Eq. (2), multiplication by r^4 and omittng the trigonometric functions, yields,

$$r^4 F_{ni}^{IV} + 2r^3 F_{ni}^{III} - r^2(1 + 2n^2 + \beta_{ci}\alpha_i^2 - r^2\alpha_i^2)\, F^{II} + r(1 + 2n^2 + \beta_{ci}\alpha_i^2$$

$$+ r^2\alpha_i^2)\, F_{ni}^{I} - F_{ni}[n^2\,(4-n^2 + \beta_{ci}\alpha_i^2 + r^2\alpha_i^2)] = 0 \tag{13}$$

where $(\quad)' = d/dr$; $\alpha_i^2 = No_i/D_i$ (see Eq. [6]); and $\beta_{c_i} = N_{oc_i}/N_{o_i}$

A power series solution, of the form given below, is assumed

$$F_{ni}(r) = \sum_{j=0}^{\infty} A_{nij}\, r^{j + s_{ni}} \tag{14}$$

Substitution into Eq. (13) yields

$$\sum_{j=0}^{\infty} A_{nij} r^{j+s_{ni}} \{ (j+s_{ni}-1)^4 - (2+2n^2 + \beta_{c_i}\alpha_i^2)\,(j + s_{ni}-1)^2 + [(n-1)^2 +$$

$$\beta_{c_i}\alpha_i^2\,(1-n^2)]\} + \sum_{j=0}^{\infty} A_{nij} r^{j+s_{ni}+2} \alpha_i^2\,[(j+1)^2 - n^2] = 0 \tag{15}$$

This equation when expanded yields terms multiplying powers of r of the form $r^{s_{ni}+j}$, with j = 0,1,...2. The multiplying terms contain A_{ni0} for j = 0; A_{ni1} for j = 1; and A_{nij} and A_{nij-2} for all other j-values (2,3,...). It can easily be shown that one solution is obtained by requiring $A_{ni0} \neq 0$ and $A_{ni1} = 0$, and another solution is obtained by requiring $A_{ni0} = 0$ and $A_{ni1} \neq 0$. Both lead to the same solution for the buckling problem.

Thus, for j = 0 the indicial equation is:

$$(s_{ni}-1)^4 - (2 + 2n^2 + \beta_{ci}\alpha_i^2)(s_{ni}-1)^2 + [(n^2-1)^2 + \beta_{c_i}\alpha_i^2(1-n^2)] = 0 \quad (16)$$

Note that this equation, Eq. (16), and all those corresponding to other values of j ($\neq$ 0) are quartic algebraic equations in s_{ni}.

The four roots of this particular quartic equation, Eq. (16), are:

$$(s_{ni}-1)^2_{1,2} = (1+n^2 + \frac{\beta_{ci}\alpha_i^2}{2}) + \sqrt{4n^2 + 2n^2\beta_{ci}\alpha_i^2 + \frac{\beta_{ci}^2\alpha_i^4}{4}} = \gamma_{ni} + \delta_{ni}^{1/2}$$

$$(s_{ni}-1)^2_{3,4} = (1+n^2 + \frac{\beta_{ci}\alpha_i^2}{2}) - \sqrt{4n^2 + 2n^2\beta_{ci}\alpha_i^2 + \frac{\beta_{ci}^2\alpha_i^4}{4}} = \gamma_{ni} - \delta_{ni}^{1/2} \quad (17)$$

where γ_{ni} is equal to the terms in parentheses, and δ_{ni} is equal to the terms under the radical sign of the right-hand side of Eqs. (17). The recurrence formula, in the general solution [for all k - see (Eq. 11)] is:

$$A_{knij} = - \frac{\alpha_i^2[(j+s_{kni}-2)^2 - n^2]\, A_{knij-2}}{[(j+s_{kni}-1)^4 - (s_{kni}-1)^4] - (2 + 2n^2 + \beta_{ci}\alpha_i^2)\, [(j+s_{kni}-1)^2(s_{kni}-1)^2]}$$

$(j = 2,4\ldots \text{ even})$ (18)

$(k = 1,2,3,4)$

The various types of solution depend on the type of roots (real, equal or complex). The general solution (for the case of four distinct roots);

$$F_{kni}(r) = \sum_{j=0}^{\infty} A_{knij}\, r^{j + s_{kni}} \qquad (k = 1,2,3,4) \tag{19}$$

where n is the wave number;

i refers to annular plate i;

k identifies the four independent solutions;

j refers to the j^{th} term; and

s_{kni} denotes the k^{th} root of the indicial equations, Eq. (16), see Eqs. (17), for plate i and wave number n.

3.1. Types of Roots, s_{kni}

The roots of the indicial equation, Eq. (16), can be either real, equal or complex conjugate. First, we'll determine the various types of the roots, and later we'll discuss the intervals of validity. In general, the roots can be written as

$$s_{kni} = s^R_{kni} + i\, s^I_{kni} \tag{20a}$$

where

$$
\begin{aligned}
s^R_{ni1} &= \mathrm{Re}\,[s_{ni}] = 1 + p^R_{ni1} \quad &:& \quad s^I_{ni1} = \mathrm{Im}\,[s_{ni}] = \rho^I_{ni1} \\
s^R_{ni2} &= \qquad\qquad = 1 - p^R_{ni1} \quad &:& \quad s^I_{ni2} = \qquad\qquad = -p^I_{ni1} \\
s^R_{ni3} &= \qquad\qquad = 1 + p^R_{ni2} \quad &:& \quad s^I_{ni3} = \qquad\qquad = p^I_{ni2} \\
s^R_{ni4} &= \qquad\qquad = 1 - p^R_{ni2} \quad &:& \quad s^I_{ni4} \qquad\qquad = -p^I_{ni2}
\end{aligned}
\tag{20b}
$$

The values of p^R_{ni1}, p^I_{ni1}, p^R_{ni2}, p^I_{ni2} depend on the values of δ_{ni} [see Eqs. (17)]. For $\delta_{ni} > 0$ the quantities in Eq. (20) are as follows: First, let

$$\phi_{ni1} = \gamma_{ni} + \delta^{1/2}_{ni} \quad : \quad \phi_{ni2} = \gamma_{ni} - \delta^{1/2}_{ni}$$

Next

$$
\begin{aligned}
&\text{if } \phi_{ni1} \geq 0, \text{ then } p^R_{ni1} = \sqrt{\phi_{ni1}} \quad : \quad p^I_{ni1} = 0 \\
&\text{if } \phi_{ni1} < 0, \text{ then } p^R_{ni1} = 0 \quad : \quad p^I_{ni1} = \sqrt{-\phi_{ni1}} \\
&\text{if } \phi_{ni2} \geq 0, \text{ then } p^R_{ni2} = \sqrt{\phi_{ni2}} \quad : \quad p^I_{ni2} = 0 \\
&\text{if } \phi_{ni2} < 0, \text{ then } p^R_{ni2} = 0 \quad : \quad p^I_{ni2} = \sqrt{-\phi_{ni2}}
\end{aligned}
\tag{21}
$$

Moreover, for $\delta_{ni} = 0$ we have,

if $\gamma_{ni} > 0$, then $p^R_{ni1} = p^R_{ni2} = \sqrt{\gamma_{ni}}$: $p^I_{ni1} = p^I_{ni2} = 0$

$$\text{if } \gamma_{ni} = 0, \text{ then } p^R_{ni1} = p^R_{ni2} = p^I_{ni1} = p^I_{ni2} = 0 \tag{22}$$

if $\gamma_{ni} < 0$, then $p^R_{ni1} = p^R_{ni2}$; $p^I_{ni1} = p^I_{ni2} = \sqrt{-\gamma_{ni}}$

and for $\delta_{ni} < 0$

$$p^R_{ni1} = - p^R_{ni2} = \sqrt{\left(\gamma_{ni} + \sqrt{\gamma_{ni} + |\delta_{ni}|}\right)/2} \tag{23}$$

$$p^I_{ni1} = p^I_{ni2} = \sqrt{-\delta_{ni}}/2/p^R_{ni1}$$

The range for the various types of roots depends on the magnitude of $\beta_{ci}\alpha_i^2$ for each of the values of δn_i.

For $\delta_{ni} > 0$, the roots are real, if $\phi_{ni2} \geq 0$ (meaning $\gamma_{ni} \geq \delta_{ni}^{1/2}$).

thus, if $\beta_{ci}\alpha_i^2 > n^2 - 1$, then $s_{ni3} \neq s_{ni4}$ (Real) ,

(24)

and if $\beta_{ci}\alpha_1^2 = n^2 - 1$, then $s_{ni3} = s_{ni4}$(Real)

The four roots are equal if $\delta_{ni} = 0$, thus

$$\beta_{ci}\alpha_i^2 = -4n\left(n - \sqrt{n^2-1}\right) \tag{25}$$

(See figs.3).

The various solutions, $F_{nik}(r)$, depend on the type of roots, (real, equal or complex). The relation between $\beta_{ci}\alpha_i^2$ and P_{cr} is

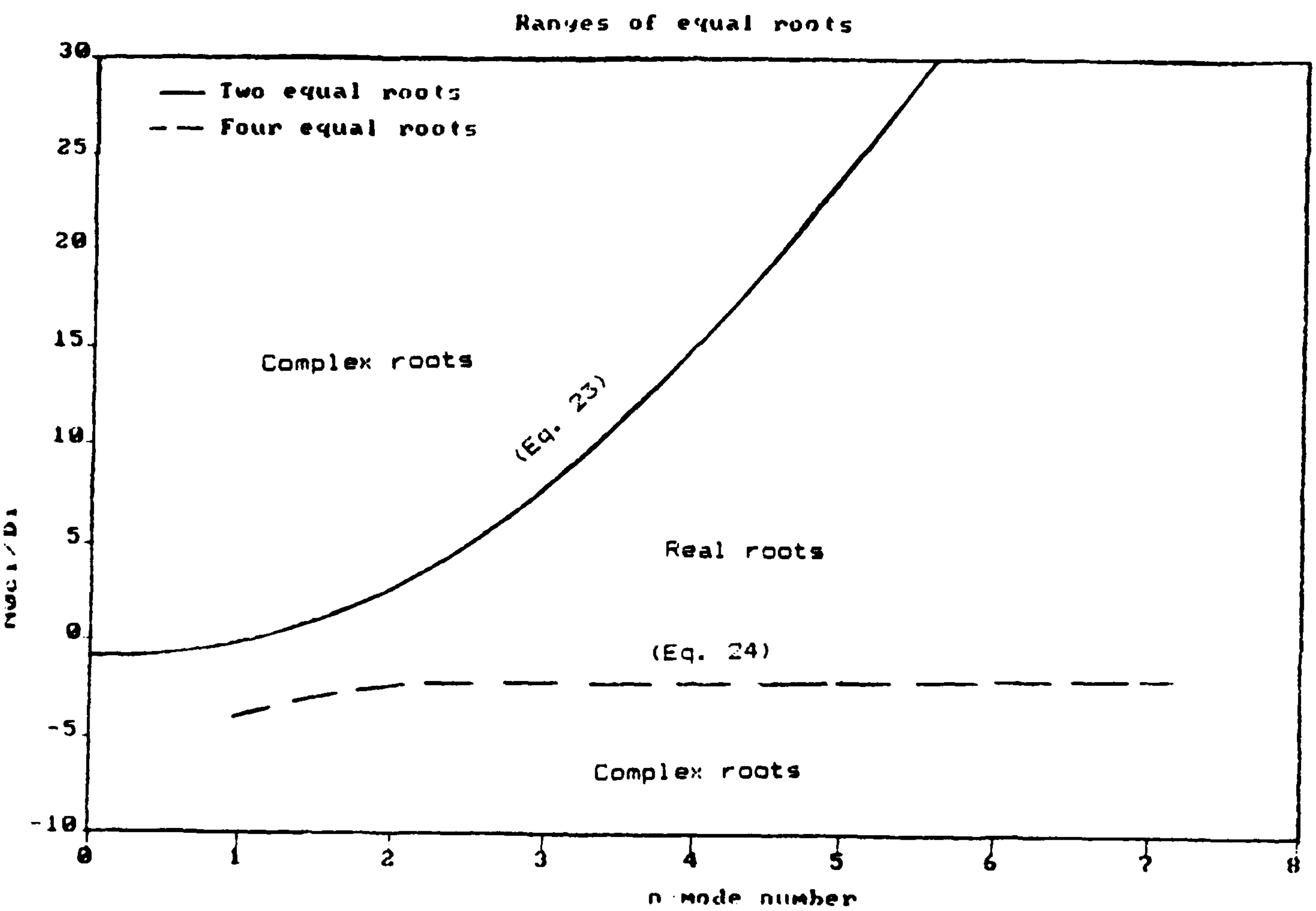
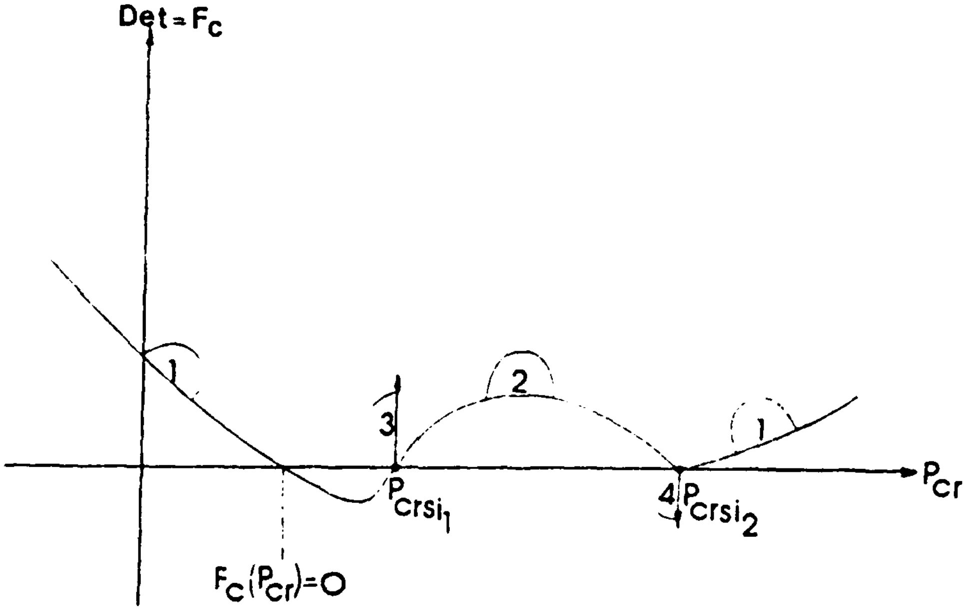


Fig. 3: Range of Equals Roots for various Buckling Modes.

$$P_{cr} = \frac{\beta_{ci}\alpha_i^2 D^o}{N_{o_{ci}}^{(1)}}$$

where $N_{o_{ci}}^{(1)}$[see Eq. (6)] is the value of $N_{o_{ci}}$ per unit of buckling force, and D^o is the relative flexural rigidity.

The two lines appearing on Fig. 3 represent points of singularity, for a given n, in the buckling characteristic function [see Figure 4]. The characteristic function is obtained by expanding the determinant that yields critical loads. This determinant results from requiring a nontrivial solution to the set of homogenous linear algebric equations in the undetermined constants, A_{kni}. This set is obtained by using the boundary and joint conditions.



1) Complex roots solution
2) Real roots solution
3) Two equal roots solution
4) Four equal roots solution

$$P_{crs_1} = (n^2 - 1)\ D_i/N_{oci}^{(1)}$$

$$P_{crsi_2} = -4n\ (n - n^2 - 1)\ Di/N_{oci}^{(1)}$$

Fig. 4. The characteristic function versus the critical forces for the various types of solution

The singularity in the characteristic function is due to the discontinuity in the type of the roots. In each interval, the function is continuous, but not at the ends of the interval with the equal roots. At these points, the function is approaching zero either from the positive or the negative side, and exactly at these points, the function can assume any value, including zero. This numerical phenomenon occurs because the solutions for complex or real roots converge to two or four equal solutions and do not approach the appropriate solutions for equal roots. The characteristic function is calculated through the determinant of the buckling matrix. The cases of equal solutions means, two or four equal rows in that matrix, meaning the determinant is zero. The roots for $n = 0$, 1 and $n \geq 2$, and for $-\infty \leq \beta_{ci} \leq \infty$ are described in Table 1.

3.2 Types of solutions, $F_{kni}(r)$

This type of solutions depends on the type of roots. In general, the power series solution based on the recurrence formula, Eq. (17), is correct, for all four independent solutions ($k = 1,2,3,4$), only if the roots are real, i.e., they are different and the difference between each of them is a non-integer.

The solution in case of two equal roots, (in general) equals:

$$F_{k+1ni}(r) = C_1 F_{kni}(r) \ln r + \sum_{j=0}^{\infty} b_{nij}\, r^{s_{kni}+j} \tag{26}$$

where F_{kni} and $F_{(k+1)ni}$ are solutions corresponding to two equal roots,

C_1 is a coefficient that can be obtained through the recurrence procedure, and b_{nij} are new undetermined coefficients.

Table I. The roots for various n and β_{ci} are: (The subscripts n_i were omitted for simplicity)

n	root	$\beta_{ci} > 0$	$\beta_{ci} = 0$	$\beta_{ci} < 0$					
0	s_1	$1+(1+\beta_{ci}a_i^2)^{1/2}$	2	$\lvert\beta_{ci}\rvert a_i^2 < 1$:	$1+(1-\lvert\beta_{ci}\rvert a_i^2)^{1/2}$	$\lvert\beta_{ci}\rvert a_i^2 = 1$:	1	$\lvert\beta_{ci}\rvert a_i^2 > 1$:	$1+i(1-\lvert\beta_{ci}\rvert a_i^2)^{1/2}$
	s_2	$1-(1+\beta_{ci}a_i^2)^{1/2}$	0	"	$1-(1-\lvert\beta_{ci}\rvert a_i^2)^{1/2}$	"	1	"	$1-i(1-\lvert\beta_{ci}\rvert a_i^2)^{1/2}$
	s_3	2	2		2		2		2
	s_4	0	0		0		0		0
1	s_1	$1+(4+\beta_{ci}a_i^2)^{1/2}$	3	$\lvert\beta_{ci}\rvert a_i^2 < 4$:	$1+(4-\lvert\beta_{ci}\rvert a_i^2)^{1/2}$	$\lvert\beta_{ci}\rvert a_i^2 = 4$:	1	$\lvert\beta_{ci}\rvert a_i^2 > 4$:	$1+i(4-\lvert\beta_{ci}\rvert a_i^2)^{1/2}$
	s_2	$1-(4+\beta_{ci}a_i^2)^{1/2}$	-1	$\lvert\beta_{ci}\rvert a_i^2 < 4$:	$1+(4-\lvert\beta_{ci}\rvert a_i^2)^{1/2}$	$\lvert\beta_{ci}\rvert a_i^2 = 4$:	1	$\lvert\beta_{ci}\rvert a_i^2 > 4$:	$1+i(4-\lvert\beta_{ci}\rvert a_i^2)^{1/2}$;
	s_3	1	1		1		1		1
	s_4	1	1		1		1		1
$n>2$		$\beta_c a_i^2 > 2.15$		$\lvert\beta_{ci}\rvert a_i^2 \le 2.15$:		$\lvert\beta_{ci}\rvert a_i^2 > 2.15$			
	s_1	$1+\gamma+\delta^{1/2}$	$2+n$		$1+\gamma+\delta^{1/2}$				$1+\gamma+i\delta^{1/2}$
	s_2	$1+\gamma+\delta^{1/2}$	$-n$		$1-\gamma+\delta^{1/2}$				$1-\gamma+i\delta^{1/2}$
	s_3	$1+\gamma-\delta^{1/2}$	$2-n$		$1+\gamma-\delta^{1/2}$				$1+\gamma-i\delta^{1/2}$
	s_4	$1+\gamma-\delta^{1/2}$	n		$1+\gamma-\delta^{1/2}$				$1-\gamma-i\delta^{1/2}$

$\gamma = 1 + n^2 + \beta_{ci}a_i^2/2$ $\qquad$ $\gamma = 1 + n^2\lvert\beta_{ci}\rvert a_i^2/2$

$\delta = (2n^2 + \beta_{ci}a_i^2/2)^2 - 4n^2(n^2-1)$ $\qquad$ $\delta = -(2n^2 - \lvert\beta_{ci}\rvert a_i^2 2)^2 + 4n^2(n^2-1)$

The solutions in case of complex roots are based on the recurrence formula, appearing in Eq. (17). The two solutions are determined by considering the real and imaginary parts as two real solutions yielding:

$$F_{i1} = r^{s^R_{ni1}}[S_{ni1} \cos (s^I_{ni1} \; \ell n\ r) - S_{ni2} \sin (s^I_{ni1} \; \ell n\ r)]$$
$$F_{i2} = r^{s^R_{ni1}}[S_{ni1} \sin (s^I_{ni1} \; \ell n\ r) + S_{ni2} \cos (s^I_{ni1} \; \ell n\ r)] \qquad (27)$$

where

$$S_{ni1} = (1 - a^R_{ni2}\ \alpha_i r^2 + \;.....(-1)^{N/2}\alpha_i^{N/2} a^R_{niN} r^N)\ r^{s^R_{ni1}}$$
$$S_{ni2} = (- a^I_{ni2}\ \alpha_i r^2 + \;.....(-1)^{N/2}\alpha_i^{N/2} a^I_{niN} r^N)\ r^{s^R_{ni1}} \qquad (27.1)$$
$$A_{nij} = a^R_{nij} + i a^I_{nij}$$

In the case of $\beta_{ci} = 0$, the roots are integer and the second solution, Eq. (26) is equal to, $Y_n(r)$, a Bessel function of the second kind of order n. Using the power series, this solution becomes:

$$F_{ni2} = C_1 F_{1ni} \ln r + C_2 r^n \ln r + \sum_{j=0}^{\infty} b_{nij}\ r^{j + s_{ni2}} \qquad (28)$$

where C_1, C_2 are constants calculated through the recurrence procedure, n is the number of full waves, and $s_{ni2} = 2-n$ is the second root of indical equations (See Table 1).

The type of solutions for the buckling equation depends on the following parameters:

(1) Number of circumferential full waves: $n = 0,1,2,\ldots N$

(2) The value of ratio of the non-uniform part to the uniform one, β_{ci} < or = or > 0.

(3) The type of roots of the indicial equation, e.g., real, complex, equal or different by an interger.

The various solutions for the parameters, mentioned above, are described in Table 2. Moreover, the various solutions are shown in Appendix B.

The solutions derived, using the power series method, usually converge. The convergence is checked either by using tests for infinite series [21] or by substitution of the solution into the buckling equations Eq. [13). The criterion for convergence using the second approach is:

$$\left| L\,(F_{kin} r_j) \right| \leq 10^{-6} \tag{29}$$

where: L is a differential operator, see Eq. (13),

r_j is the radial coordinate at the boundaries of an annular plate, and

$F_{kin}(r_j)$ is the k-th solution of plate i different from zero.

If the test fails, then the number of terms in this series must be increased. The results obtained by using these tests are considered to be accurate, when compared to the results reported in the open literature (used as benchmarks).

TABLE 2: Solutions for $\beta_{ci} \geq 0$

(Subscript are omitted for simplicity)

n	root	Solution
0	$s_1=1+\phi_i$	$F_1 = r^{1+\phi_i}\{ \frac{1}{1+\phi_i} - \frac{u^2}{[(2+\phi_i)^2 - \phi_i^2](3+\phi_i)} + \ldots\ldots + \frac{(-1)^{N/2}u^N}{(N+1+\phi_i)\prod_{j=2}^{N}[(j+\phi_i)^2-\phi_i^2]} \}$
	$s_2=1-\phi_i$	$F_2 = r^{1-\phi_i}\{ \frac{1}{1-\phi_i} - \frac{u^2}{[(2-\phi_i)^2 - \phi_i^2](3-\phi_i)} + \ldots\ldots + \frac{(-1)^{N/2}u^N}{(N+1-\phi_i)\prod_{j=2}^{N}[(j-\phi_i)^2-\phi_i^2]} \}$
	$s_3=2$	$F_3 = r^2\{ \frac{1}{2} - \frac{u^2}{(3^2-\phi_i^2)4} + \ldots\ldots + \frac{(-1)^{N/2}u^N}{(N+2)\prod_{j=2}^{N}[(j+1)^2-\phi_i^2]} \}$
	$s_4=0$	$F_4=C$

Notes: $j=0,2,4\ldots$ even

$\phi_i = (1+\beta_{ci}\alpha_i^2)^{1/2}$

$u=\alpha_i r$

TABLE 2: Solutions for $\beta_{ci} > 0$ (continued)

n	root	Solution
1	$s_1=1+\varnothing_i$	$F_1= r^{1+\varnothing_i}\{ \frac{1}{1+\varnothing_i} - \frac{u^2}{[(2+\varnothing_i)^2-\varnothing_i^2](3+\varnothing_i)} + \ldots + \frac{(-1)^{N/2}u^N}{(N+1+\varnothing_i)\prod_{j=2}^{N}[(j+\varnothing_i)^2-\varnothing_i^2]}\}$
	$s_2=1-\varnothing_i$	$F_2= r^{1-\varnothing_i}\{ \frac{1}{1-\varnothing_i} - \frac{u^2}{[(2-\varnothing_i)^2-\varnothing_i^2](3-\varnothing_i)} + \ldots + \frac{(-1)^{N/2}u^N}{(N+1-\varnothing_i)\prod_{j=2}^{N}[(j-\varnothing_i)^2-\varnothing_i^2]}\}$
	$s_3=1$	$F_3=r$
	$s_4=1$	$F_4= r\ln(r)+ \sum_{j=2}^{N} b_j r^{j+i}$ where: $b_2=\alpha_i^2/(2^2-\varnothing_i^2)/2.$ $b_j=-\frac{\alpha_i^2[(j+1)^2-n^2]b_{j-2}}{(j^2-\varnothing_i^2)j^2}$

Notes: $j=0,2,4....$even

$\varnothing_i=\mathrm{sqr}(4+\beta_{ci}\alpha_i^2)$

$u=\alpha_i r$

TABLE 2: Solutions for $\beta_{ci} > 0$ (continued)

n	root	Solution
$>=2$	$s_1=1+\varnothing_1$	$F_1 = r^{1+\varnothing_1}\{1 - \frac{u^2[(1+\varnothing_1)^2-n^2]}{[(2+\varnothing_1)^2-\varnothing_1^2][(2+\varnothing_1)^2+\varnothing_2^2]}$ $\ldots + \frac{(-1)^{N/2}u^N\prod_{j=2}^{N}((j+\varnothing_1-1)^2-n^2)}{\prod_{j=2}^{N}[(j+\varnothing_1)^2+\varnothing_2^2][(j+\varnothing_1)^2-\varnothing_1^2]}\}$
	$s_2=1-\varnothing_1$	$F_2 = r^{1-\varnothing_1}\{1 - \frac{u^2[(1-\varnothing_1)^2-n^2]}{[(2-\varnothing_1)^2-\varnothing_1^2][(2-\varnothing_1)^2+\varnothing_2^2]}$ $\ldots + \frac{(-1)^{N/2}u^N\prod_{j=2}^{N}((j-\varnothing_1-1)^2-n^2)}{\prod_{j=2}^{N}[(j-\varnothing_1)^2+\varnothing_2^2][(j-\varnothing_1)^2-\varnothing_1^2]}\}$
	$s_3=1+i\varnothing_2$ $s_4=1-i\varnothing_2$	$F_3=r\cos[\varnothing_2\ln(r)]S_1-r\sin[\varnothing_2\ln(r)]S_2$ $F_4=r\cos[\varnothing_2\ln(r)]S_2+r\sin[\varnothing_2\ln(r)]S_1$ $S_1=-A_2\cos\Theta_2u^2+\ldots(-1)^{N/2}u^N(\prod_{j=2}^{N}A_j)\cos(\sum_{j=2}^{N}\Theta_j)$ $S_2=-A_2\sin\Theta_2u^2+\ldots(-1)^{N/2}u^N(\prod_{j=2}^{N}A_j)\sin(\sum_{j=2}^{N}\Theta_j)$ where: $\varnothing_1=1+n^2+\beta_{ci}\alpha_i^2 + [(2n^2+\beta_{ci}\alpha_i^2/2)^2\ 4n^2(n^2-1)]^{1/2}=$ $=\gamma+\delta^{1/2}$ $\varnothing_2=1+n^2+\beta_{ci}\ {}_i^2 + [-(2n^2+\beta_{ci}\ {}^2/2)^2+4n^2(n^2-1)]^{1/2}=$ $=\gamma+i\delta^{1/2}$ $A_j=sqr(R_j^2+I_j^2);\quad \Theta_j=arctg\ (I_j/R_j)$ $R_j=(j^2(j^2-4\varnothing_2^2-2\delta^{1/2})[n^2+(j-1)^2-\varnothing_2^2]+8j\varnothing_2^2(j-1)(j^2+$ $-\delta^{1/2}))/\lvert D_j\rvert^2$ $I_j=\varnothing_2[2j^2(j-1)(j^2-4\varnothing_2^2-2\delta^{1/2})-4j(j^2-\delta^{1/2})[n^2+$ $(j+1)^2-\varnothing_2^2])/\lvert D_j\rvert^2$ $\lvert D_j\rvert^2=16j^2(j^2-\delta^{1/2})^2+j^4(j^2-4\varnothing_2^2-2\delta^{1/2})^2$

TABLE 3: $B_{ci}=0$

n	root	Solution
0	$s_1=2$	$F_1=1-\frac{u^2}{2^2}+\ldots\ldots+\frac{(-1)^{N/2}u^N}{\prod_{j=2}^{N}(j+2)^2} = J_0(u)$
	$s_2=2$	$F_2=\left.\frac{\partial F_1(s)}{\partial s}\right\|_{s=s_1} = Y_0(u)$
	$s_3=0$	$F_3=$Const.
	$s_4=0$	$F_4=\ln(r)$

Notes:

$J_0(u)$ - Bessel function of first kind and of order 0.

$Y_0(u)$ - Bessel function of second kind and of order 0.

$u=\alpha_i r$

$n>=1$	$s_1=2+n$	$F_1=r^n\{1-\frac{u^2}{(2+n)^2-n^2}+\ldots+\frac{(-1)^{N/2}u^N}{\prod_{j=2}^{N}[(j+2+n)^2-n^2]}\}= J_n(u)$
	$s_2=2-n$	$F_2=\left.\frac{\partial F_1(s)}{\partial s}\right\|_{s=s_1} = Y_n(u)$
	$s_3=n$	$F_3=r^n$
	$s_4=-n$	$F_4=r^{-n}$

Notes:

$J_n(u)$ - Bessel function of first kind and of order n.

$Y_n(u)$ - Bessel function of second kind and of order n.

TABLE 4: Solutions for $\beta_{ci} \leq 0$

(Subscript are omitted for simplicity)

n	root	Solution
0	$s_1=1+\emptyset_i$	$\emptyset_i=\mathrm{sqr}[1-\mathrm{abs}(\beta_{ci})\alpha_i^2]$
$\lvert\beta_{ci}\rvert\alpha_i^2<1$	$s_2=1-\emptyset_i$	Solutions are the same as in table 2, but
	$s_3=2$	with a different $\emptyset_i$.
	$s_4=0$	
$\lvert\beta_{ci}\rvert\alpha_i^2=1$	$s_1=1$	$F_1=r$
	$s_2=1$	$F_2=r\ln(r)+\sum_{j=2}^{N} b_j r^{j+1}$
		where: $b_2=-2$
		$b_j=-\dfrac{-2-b_{j-2}\alpha_i^2 A_{nom}}{D_{en}}$
		$A_{nom}=j-1-n^2;\ D_{en}=j^2(j^2-1)$
	$s_2=2$	see table 2 with n=0
	$s_4=0$	see table 2 with n=0
$\lvert\beta_{ci}\rvert\alpha_i^2>1$	$s_1=1+i\emptyset_i$	$F_3=r\cos[\emptyset_i\ln(r)]S_1-r\sin[\emptyset_i\ln(r)]S_2$
	$s_2=1-i\emptyset_i$	$F_4=r\cos[\emptyset_i\ln(r)]S_2+r\sin[\emptyset_i\ln(r)]S_1$
		$S_1=-A_2\cos\theta_2 u^2+\ldots(-1)^{N/2}u^N(\prod_{j=2}^{N}A_j)\cos(\sum_{j=2}^{N}\theta_j)$
		$S_2=-A_2\sin\theta_2 u^2+\ldots(-1)^{N/2}u^N(\prod_{j=2}^{N}A_j)\sin(\sum_{j=2}^{N}\theta_j)$
		$\emptyset_i=(1-\lvert\beta_{ci}\rvert\alpha_i^2)^{1/2};\ A_j=(R_j^2+I_j^2)\ ;\ \theta_j=\mathrm{arctg}(I_j/R_j)$
		$R_j=\{(j-1)[j^2(j+1)-2j\emptyset_i^2]+\emptyset_i^2(3j+2)\}/\lvert D_i\rvert^2$
		$I_j=\{j^2(j+1)-2j\emptyset_i-(j-1)(3j+2)\}/\lvert D_i\rvert^2$
		$\lvert D_i\rvert^2=[j^2(j+1)-2j\emptyset_i^2]^2+[\emptyset_i^2 j^2(3j+2)^2]^2$

TABLE 4: Solutions for $\beta_{ci}<0$ (continued)

n	root	Solution
$\lvert\beta_{ci}\rvert\alpha_i^2>1$	$s_3=2$ $s_4=0$	same as in table 2 with n=0 same as in table 2 with n=0
1 $\lvert\beta_{ci}\rvert\alpha_i^2<4$	$s_1=1+\emptyset_i$ $s_2=1-\emptyset_i$ $s_3=1$ $s_4=1$	Same as in Table 2 with n=1. $\emptyset_i=sqr[1-abs(\beta_{ci})\alpha_i^2]$ $F_3=r$ $F_4=r\ln(r)+\sum_{j=0}^{N} b_j r^{j+1}$ The series coefficients are determined using the procedure described in Appendix B.2.3.1
$\lvert\beta_{ci}\rvert\alpha_i^2=4$	$s_1=1$ $s_2=1$ $s_3=1$ $s_4=1$	$F_1=r$ $F_2=r\ln(r)+\sum_{j=0}^{N} b_j r^{j+1}$ $F_3=r[\ln(r)]^2+\sum_{j=0}^{N} c_j r^{j+1}$ $F_4=r[\ln(r)]^3+\sum_{j=0}^{N} d_j r^{j+1}$ The series coefficients are determined using the procedure described in Appendix B.2.3.1
$\lvert\beta_{ci}\rvert\alpha_i^2>4$	$s_1=1+i\emptyset_i$ $s_2=1-i\emptyset_i$	$F_1=r\cos[\emptyset_i\ln(r)]S_1-r\sin[\emptyset_i\ln(r)]S_2$ $F_2=r\cos[\emptyset_i\ln(r)]S_2+r\sin[\emptyset_i\ln(r)]S_1$ $S_1=-A_2\cos\theta_2 u^2+\ldots(-1)^{N/2}u^N(\prod_{j=2}^{N}A_j)\cos(\sum_{j=2}^{N}\theta_j)$ $S_2=-A_2\sin\theta_2 u^2+\ldots(-1)^{N/2}u^N(\prod_{j=2}^{N}A_j)\sin(\sum_{j=2}^{N}\theta_j)$ $\emptyset_i=(4-\lvert\beta_{ci}\rvert\alpha_i^2)^{1/2}$; $A_j=(R_j^2+I_j^2)^{1/2}$; $\theta_j=arctg(I_j/R_j)$ $R_j=[(j-2)(j-2\emptyset_i^2)+\emptyset_i^2]/\lvert D_i\rvert^2$ $I_j=2(1-\emptyset_i^2)/\lvert D_i\rvert^2$

TABLE 4: Solutions for $\beta_{ci} < 0$ (continued)

n	root	Solution
		$\vert D \vert_i^2 = j^2[(j-2\emptyset_i^2) + \emptyset_i^2]$
	$s_3=1$	$F_3=r$
	$s_4=1$	$F_4=r\ln(r)+\prod_{j=0}^{N} b_j r^{j+1}$
		The series coefficients are determined using the procedure described in Appendix B.2.3.1
$n>=2$ $\vert\beta_{ci}\vert_i^2<2$	$s_1=1+\emptyset_1$	The same as in Table 2 with n>1.
	$s_2=1-\emptyset_1$	The same as in Table 2 with n>1.
	$s_3=1+\emptyset_2$	The same as in Table 2 with n>1.
	$s_4=1-\emptyset_2$	The same as in Table 2 with n>1.

Notes:

$$\emptyset_1=[1+n^2+abs(\beta_{ci})\alpha_i^2/2]+[4n^2+2n^2abs(\beta_{ci})\alpha_i^2+(abs(\beta_{ci})\alpha_i^2)^2/4]^{1/2}$$

$$\emptyset_2=[1+n^2+abs(\beta_{ci})\alpha_i^2/2]-[4n^2+2n^2abs(\beta_{ci})\alpha_i^2+(abs(\beta_{ci})\alpha_i^2)^2/4]^{1/2}$$

n	root	Solution
$n>=2$	$s_1=1+\emptyset_R+i\emptyset_i$	$F_1=r^{1+\emptyset_R}\{\cos[\emptyset_I\ln(r)]S_1-\sin[\emptyset_I\ln(r)]S_2\}$
	$s_2=1+\emptyset_R-i\emptyset_i$	$F_2=r^{1+\emptyset_R}\{\cos[\emptyset_I\ln(r)]S_2+\sin[\emptyset_I\ln(r)]S_1\}$
$\vert\beta_{ci}\vert_i^2>2$		$S_1=-A_2\cos\theta_2 u^2+\ldots(-1)^{N/2}u^N(\prod_{j=2}^{N}A_j)\cos(\sum_{j=2}^{N}\theta_j)$
		$S_2=-A_2\sin\theta_2 u^2+\ldots(-1)^{N/2}u^N(\prod_{j=2}^{N}A_j)\sin(\sum_{j=2}^{N}\theta_j)$
		$A_j=(R_j+I_j)^{1/2}$; $\theta_j=\mathrm{arctg}(I_j/R_j)$

Notes:

$$R_j=\frac{R_N R_D+I_N I_D}{abs(D\ )^2}\ ; \qquad I_j=\emptyset_I\frac{(I_N R_D-I_N I_D)}{abs(D\ )^2}$$

$$R_N=2(j-1)\emptyset_R+\Gamma+(j-1)^2-n^2$$

$$R_D=j^4+4j(j^2-\Gamma)\emptyset_R+2(3j^2-\Gamma)\ -4j(\Gamma-2\emptyset_I^2)\emptyset_R+\Gamma^2-4\delta+\Gamma(\Gamma^2-2j^2)+\delta$$

$$I_N=\emptyset_I[2(j-1)+2\emptyset_R]$$

TABLE 4: Solutions for $\beta_{ci}<0$ (continued)

n	root	Solution
Notes (continued):		
$I_D=\emptyset_I[4j(j^2-\gamma)+2(3j^2-\gamma)\emptyset_R+4j(\gamma+2\emptyset_R^2)+4\emptyset_R\gamma]$		
$abs(D.)^2=R_D^2+I_D^2\emptyset_I^2$		
$\gamma=1+n^2+abs(\beta_{ci})\alpha_i^2/2$		
$\delta=4n^2+2n^2 abs(\beta_{ci})\ _i^2+(abs(\beta_{ci})\ _i^2)^2/4$		
$R_s=(\gamma^2+\delta)$: $\emptyset_R=R_s\cos(\theta_s)^{1/2}$: $\emptyset_I=R_s\sin(\theta_s)$: $\theta_s=artg(\delta^{1/2}/\gamma)$		
$n>=2$ $\lvert\beta_{ci}\rvert\ _i^2>2$	$s_3=1-\emptyset_R-i\emptyset_I$ $s_4=1-\emptyset_R+i\emptyset_I$	$F_3=r^{1-\emptyset_R}\{\cos[\emptyset_I\ln(r)]S_1+\sin[\emptyset_I\ln(r)]S_2\}$ $F_4=r^{1-\emptyset_R}\{\cos[\emptyset_I\ln(r)]S_2-\sin[\emptyset_I\ln(r)]S_1\}$ The varios functions appearing in the solutions are the same as above but with $\emptyset_R=-\emptyset_R$.

TABLE 5: Solutions for $\beta_{ci} \neq 0$ and $\alpha_i = 0$

n	root	Solution	
0	$s_1=1+\emptyset_i$	$F_1=r^{(1+\emptyset_i)}$	
$\beta_{ci}>0$		$\emptyset_i=(1+\beta_{ci})^{1/2}$	
	$s_2=1+\emptyset_i$	$F_2=r^{(1-\emptyset_i)}$	
	$s_1=1+i\emptyset_i$	$F_1=r\cos[\emptyset_i \ln(r)]$	
		$\emptyset_i=i(\lvert\beta_{ci}\rvert-1)^{1/2}$: $\lvert\beta_{ci}\rvert>1$	
	$s_2=1+i\emptyset_i$	$F_2=r\sin[\emptyset_i \ln(r)]$	
$\beta_{ci}<0$	$s_1=1$	$F_1=r$	
		$\lvert\beta_{ci}\rvert=1$: $\emptyset_i=0$	
	$s_2=1$	$F_2=r\ln(r)$	
	$s_3=2$	$F_3=r^2$	
	$s_4=0$	$F_4=\text{Const.}$	
		For $n^2-1>\beta_{ci}$	For $n^2-1<\beta_{ci}$
	$s_1=1+\emptyset_1$	$F_1=r^{1+\emptyset_1}$	
	$s_2=1-\emptyset_1$	$F_2=r^{1-\emptyset_1}$	
$n>=1$	$s_3=1+\emptyset_2$	$F_3=r^{1+\emptyset_2}$	$F_3=r\cos[\emptyset_2 \ln(r)]$
	$s_4=1-\emptyset_2$	$F_4=r^{1-\emptyset_2}$	$F_4=r\sin[\emptyset_2 \ln(r)]$
		For $n^2-1=\beta_{ci}$:	
	$s_1=1+\emptyset_1$	$F_1=r^{1+\emptyset_1}$	

TABLE 5: Solutions for $\beta_{ci}=0$ and $\alpha_i=0$ (continued)

n	root	Solution
	$s_2=1-\emptyset_1$	$F_2=r^{1-\emptyset_i}$.
	$s_3=1$	$F_3=r$
	$s_4=1$	$F_4=r\ln(r)$

Notes:

$$\emptyset_1=(1+n^2+\beta_{ci}/2)+(4n^2+2n^2\beta_{ci}+\beta_{ci}^2/4)^{1/2}$$

$$\emptyset_2=(1+n^2+\beta_{ci}/2)-(4n^2+2n^2\beta_{ci}+\beta_{ci}^2/4)^{1/2}$$

$\beta_{ci}<0$	The solutions are the same as above but with $\lvert\beta_{ci}\rvert$ instead of β_{ci}.

4. NUMERICAL ANALYSIS AND RESULTS

A computer program was developed for the analysis of multi-annular circular plates, for various boundary conditions and types of loading. The program uses double precision and the maximum number of terms in each series is eighty. The "differential equation" convergence condition [(Eq. 29)] is checked for each independent solution, $F_{kni}(r)$, at each coordinate of the adjoining boundaries, r_j. The various solutions are obtained through the following four subroutines.

a) Solution for uniform stress field, meaning $\beta_{ci} = 0$.

b) Solution for real roots, but distinctly different.

c) Solution for two or more equal roots.

d) Solution for complex roots.

A relation exists between the various subroutines. The solutions obtained using subroutine (b), for integer roots, converges to the one determined by using subroutine (a). The solution for nearly equal roots, obtained by subroutine (b) for real roots, or subroutine (d) for complex roots with equal real parts and very small imaginary parts, do not converge to the solutions obtained by subroutine (c).

Numerical results are generated and discussed for a two part plate, annular and circular (inner), with constant thickness only. The main parameter influencing the buckling force are:

1) The ratio of the radii r_i/r_o, inner radius to outer one.

2) The ratio of the moduli of elasticity, E_2/E_1, inner part.

3) The number of waves in the circumferential direction (buckling mode).

4) The location of the loading, applied at the outer radius as opposed to the inner one.

5) The boundary conditions.

In order to get an understanding of the influence of the various parameters, a few cases are analyzed, and some simple parametric studies are performed.

4.1: Example No. 1 (Simply-supported)

The first example is a two-part plate loaded and supported at the outer boundary, see Fig. 5. Critical (buckling) loads are calculated for the following parameters:

$\nu_1 = \nu_2 = 1/3$ (n= 0,1 modes)

$t_1 = t_2 = 1.0$

$E_2/E_1 = 0$ (Hole), 0.01 (cir. part very flexible),

0.1, 0.5, 1 (uniform),

2, 10, 100 (cir. part very stiff)

$r_1/r_0 = 0.0, 0.1$ .... 0.9 (Dec. = 0.1)

The analysis using the above data covers the whole spectrum of rigidity ratios starting with a single annular plate ($E_2/E_1 = 0$), through a plate with a very flexible central part $E_2/E_1 = 0.01$ to a plate with very rigid inclusion ($E_2/E_1 = 100$).

Curves of critical loads versus the ratio of the radii, r_1/r_0, for different (E_2/E_1) values for n = 0 (axisymmetric) and n = 1, with $\nu = 1/3$ are presented on Fig. 5 and 6, respectively. From these two figures the following conclusions can be drawn:

1) The lowest critical load corresponds to the first mode, even for large values of E_2/E_1.

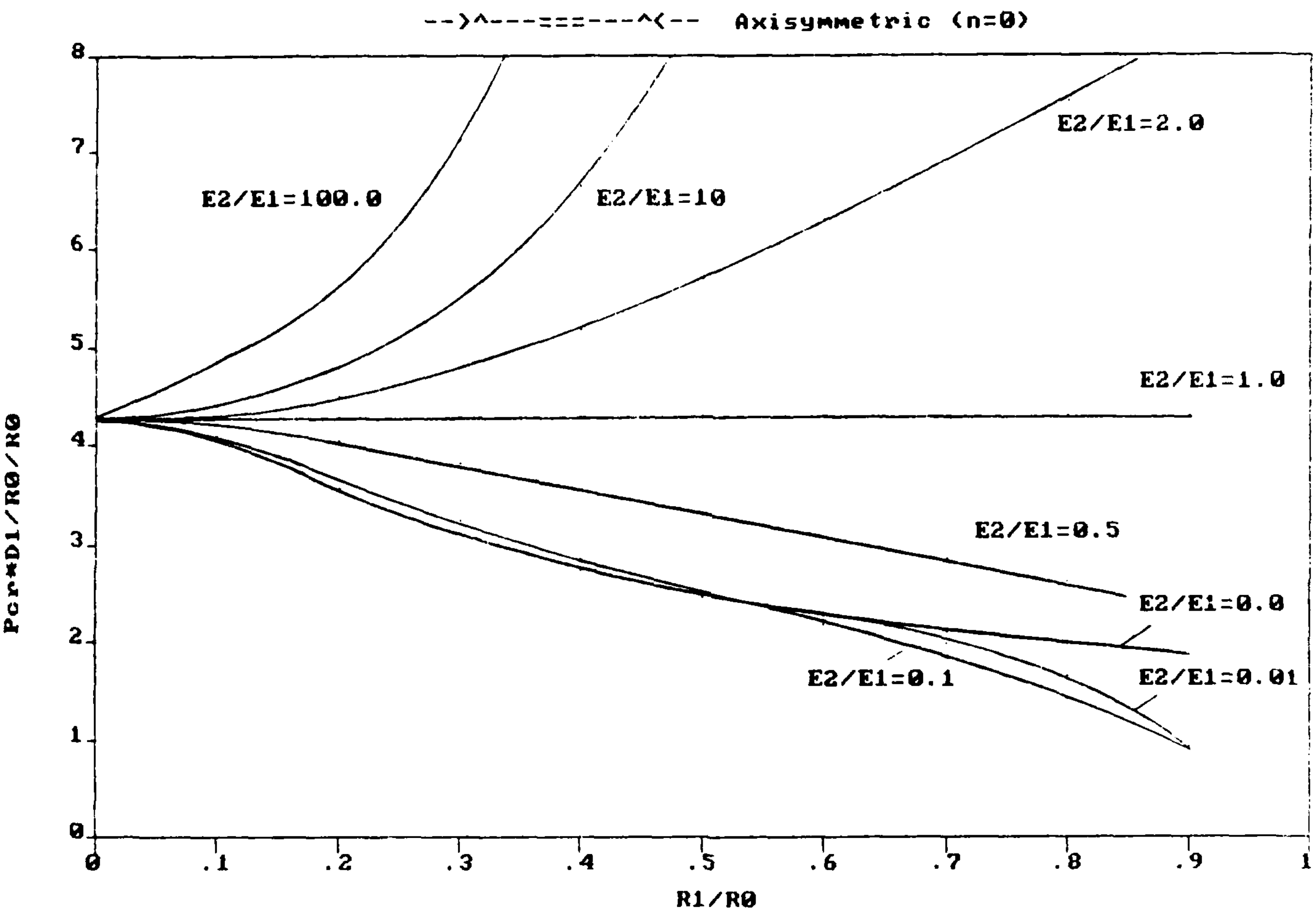


Fig. 5. The buckling force versus the radii ratio (r_1/r_o) for various rigidity ratios, with n = 0 (axisymmetric mode) and a simply-supported plate. The plate is loaded at the outer boundary.

2) The critical load decreases with increasing value of r_1/r_o for $E_2/E_1 <$ 1.0. On the contrary, for $E_2/E_1 > 1.0$ the critical load increases with increasing value for r_1/r_0.

3) For stiff inclusions ($E_2/E_1 > 1$), the buckling mode is such that both parts of the composite plate participate (the inner complete circular plate and the annular part), and buckling seems to be triggered by the annular part.

4) For flexible inclusions ($E_2/E_1 < 1$), the buckling mode depends on the values of both ratios, E_2/E_1 and r_1/r_0. For large values of E_2/E_1 (definitely for $0.5 \leq E_2/E_1 < 1$), the mode behavior is similar to that corresponding to the case of $E_2/E_1 > 1$ (See 3) above). On the contrary, for extremely small values of E_2/E_1 (it is checked for values of 0.1 and 0.01 and the observation probably is true for some E_2/E_1 values in the range of $0.1 < E_2/E_1 < 0.5$) the mode behavior depends on the ratio of the radii (r_1/r_0). For $E_2/E_1 = 0.1$, as long as r_1/r_0 is smaller than approximately 0.55, the buckling mode seems to be triggered primarily by the annular part and the critical load is a little larger than the one corresponding to $E_2/E_1 = 0$ (pure annular plate). On the other hand for r_1/r_0 values larger than 0.55, the critical load is smaller than that of the pure annular plate and the buckling mode is triggered by the inner and more flexible part. It appears that for $E_2/E_1 = 0.1$, the corresponding curve approaches the value of 0.4282 as r_1/r_0 approaches unity (a complete circular plate with bending rigidity equal to one tenth of that of the case $E_2 = E_1$).

5) The graph for a single annular plate ($E_2/E_1 = 0$), is similar to the one appearing in Meissner's paper [10]. The only difference is in the result for the homogeneous plate ($E_2/E_1 = 1$). The present result for the

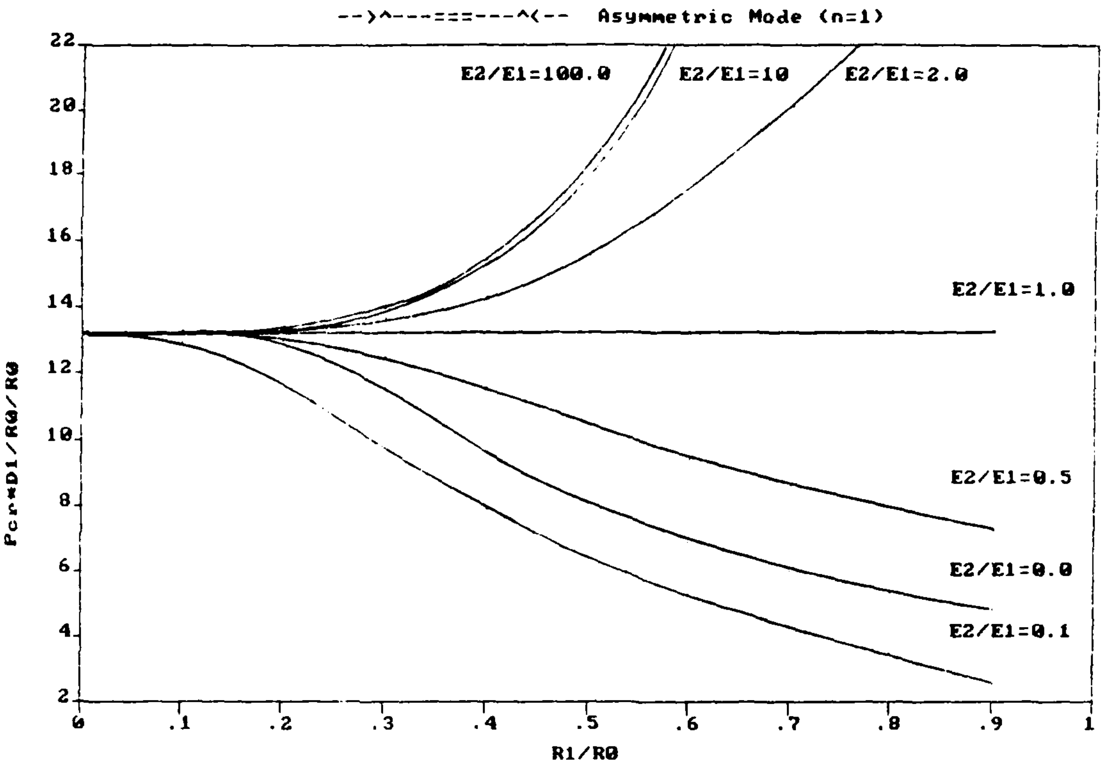


Fig. 6: P_{cr} versus r_1/r_o for various E_2/E_1, with n = 1 (asymmetric mode) and simply supported plate. The plate is loaded at the outer boundary.

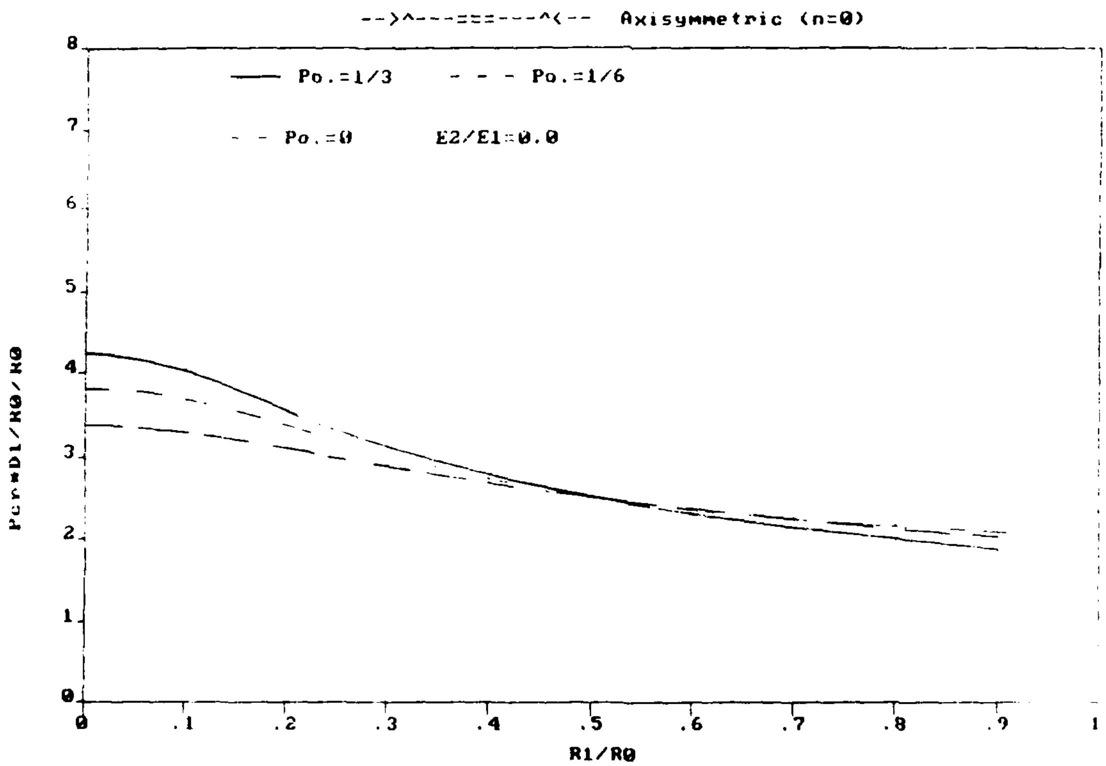
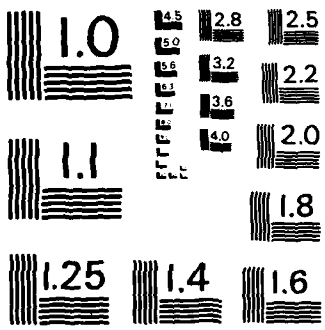


Fig. 7. The influence of the Poisson ratio on the axisymmetric buckling force for a simply-supported annular plate loaded at the outer boundary.

AD-A162 371 BUCKLING OF DELAMINATED SHELLS AND MULTI-ANNULAR PLATES 2/2
(U) GEORGIA INST OF TECH ATLANTA SCHOOL OF ENGINEERING
SCIENCE AN.. G J SIMITSES ET AL. OCT 85

UNCLASSIFIED AFOSR-TR-85-1081 AFOSR-83-0243 F/G 20/11 NL



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

critical load is $4.282D/r^2$ and not $4.20D/r^2$ as predicted by Meissmer. The present result corresponds to $\nu = 0.333$ and Meissmer's to $\nu = 0.3$. Thus, the difference is a measure of the Poisson's ratio effect.

It appears that the influence of the Poisson's ratio is significant for annular plates with either small or large values of the ratio for radii. For small values ($r_1/r_o < 0.4$), the higher the Poisson's ratio the stronger the configuration (against buckling). For $r_1/r_o = 0$ (full circular plate), the difference between the critical loads, corresponding to $\nu = 0.333$ and $\nu = 0$, is of the order of over 25%. This difference diminishes (for the annular plate) as r_1/r_0 approaches the value of 0.5. For higher values of r_1/r_0, the influence reverses direction and it increases with increasing ratio values. Note that for $r_1/r_0 = 0.9$, the critical load for $\nu = 0$ is higher than that for $\nu = 0.333$ by approximately 10%.

4.2 Example No. 2 (clamped)

In the second example, the influence of the boundary conditions is checked, through the analysis of a clamped plate.

In this example, the clamped plate is loaded at the external boundary. The critical load curves for a single annular plate are presented on Fig. 8 and for the two-part plate on Fig. 9. The results are determined for the following parameters:

$\nu_1 = \nu_2 = 1/3$

$t_1 = t_2 = 1.0$

$n = 0,1,2,3,4$ (wave number)

$E_2/E_1 = 0,0.1,0.5,1,2,10,100$ $(n = 0)$

$r_1/r_o = 0,0.1$ [illegible] , 0.8

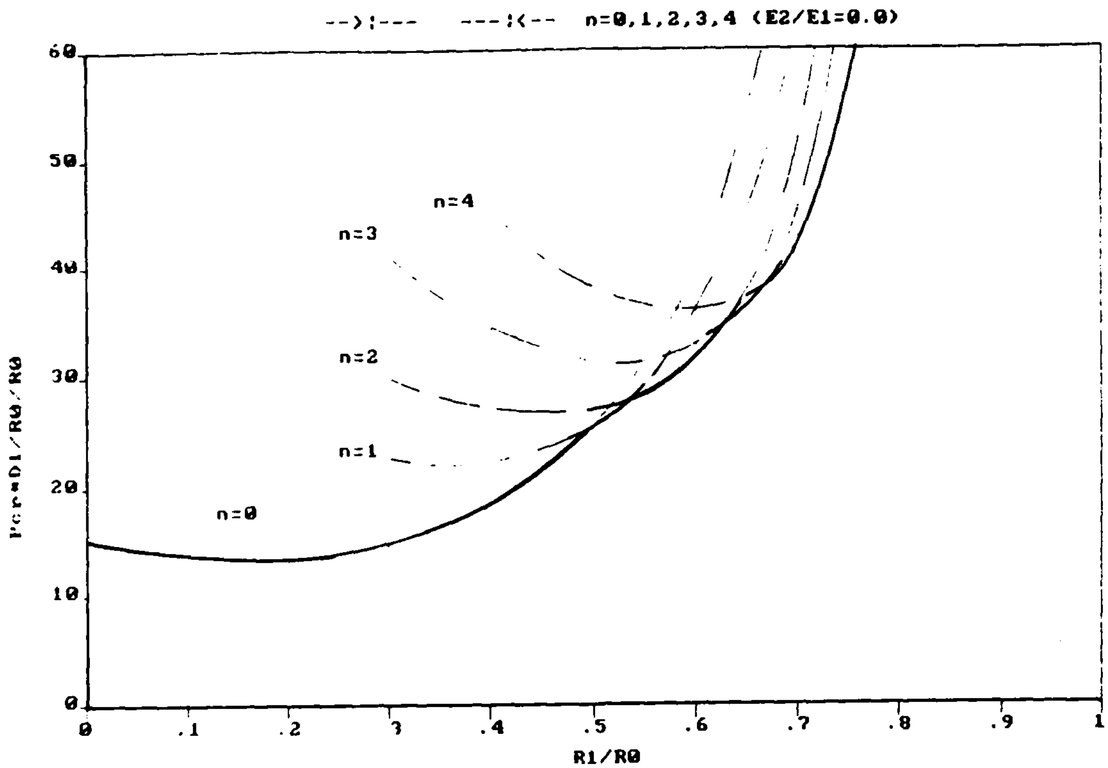


Fig. 8. The smallest buckling force of a clamped annular plate, loaded at the outer edge.

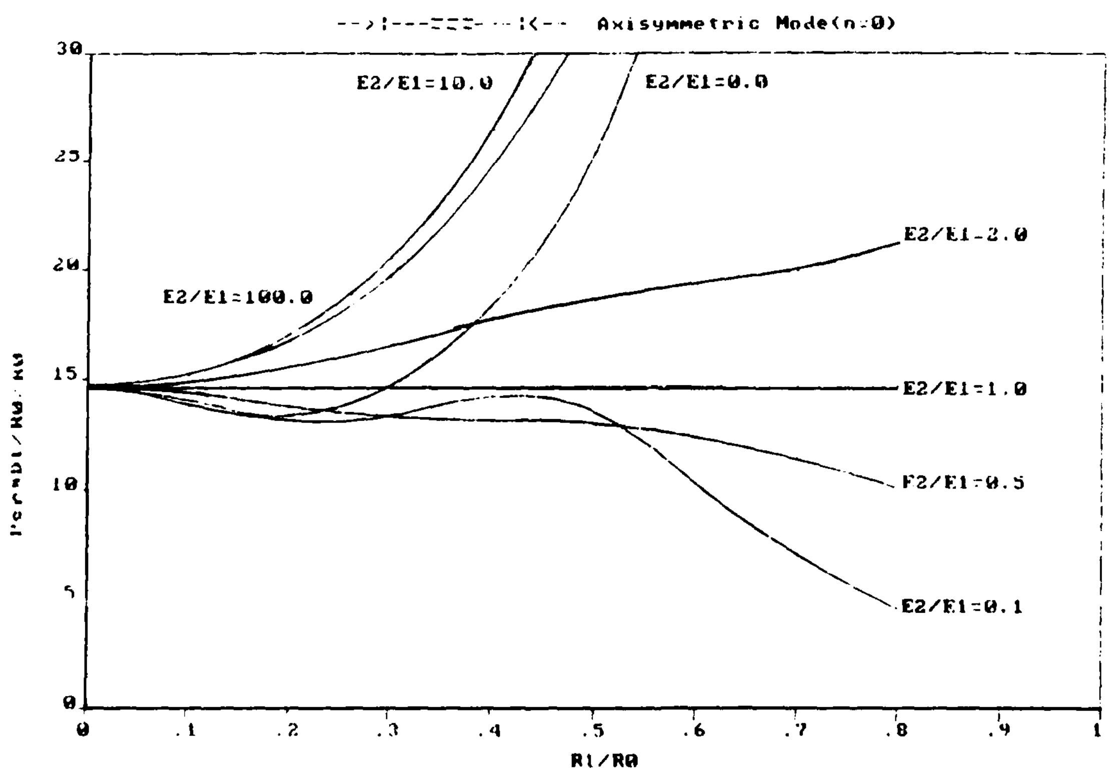


Fig. 9: P_{cr} versus r_1/r_o for various rigidity ratios with n = 0, clamped and loaded at the outer boundary.

The critical load curves versus the ratio of the radii, for a single annular and for wave numbers, zero to four, are presented on Fig. 8. The curve for the axisymmetric mode is exactly the same as the one appearing in Meissmer's paper [10]. The results appearing in Ref. [10] are limited to $r_1/r_0 \leq 0.563$. For ratios greater than 0.52 the higher buckling modes govern buckling which is in contrast to the previous example (simply supported case). On Fig. 9, the critical load curves, corresponding to different E_2/E_1-values, n = 0 (anisymmetric), and $\nu = 1/3$ are presented. In this case, Fig. 9, the following conclusions can be drawn.

1) For $E_2/E_1 > 1$ the critical load increases with increasing r_1/r_0 - values, while for $E_2/E_1 < 1$ the picture is a little more complex. First of all, spot checking of these results ($E_2/E_1 > 1$) showed that the critical condition is governed by axisymmetric behavior (n = 0). It is also observed that, as the middle section (the complete circle) becomes stiffer the critical load increases more rapidly with increasing r_1/r_0 -values.

2) For $E_2/E_1 < 1$, we observe several things. First, as long as the stiffness of the middle section is only two to three times or less smaller than the stiffness of the annular (outer) section the response is similar to that of the simply supported case, and there is a decrease in value of the critical load as the r_1/r_0-value increases. In this case the buckling mode is such that both parts (inner and outer) participate equally.

On the other hand, when E_2/E_1 is zero, we have already seen (Fig. 8) that the critical load increases as r_1/r_0 increases, and the mode becomes non-axisymmetric. For E_2/E_1 very small, or for inner section stiffnesses several times smaller than the outer section stiffness, it

is observed that for low values of the r_1/r_0 ratio the behavior is similar to that of $E_2/E_1 = 0$. This means that there is initially a decrease of the critical load as r_1/r_0 increases; a minimum value is reached; and then there is an increase (see curve corresponding to $E_2/E_1 = 0.1$). As the value of r_1/r_0 further increases, there is a distinct decrease in the critical load. The explanation here is that in the former case buckling is governed primarily by the annular (outer) section while for the latter case buckling is governed by the inner and less stiff part. Clearly in both cases (for the full range of r_1/r_0-values), the presence of a very flexible inner part weakens the annular plate. Note that the curve corresponding to $E_2/E_1 = 0$ is higher than that corresponding to $E_2/E_1 = 0.1$.

4.3 Example No. 3 (Load at $r = r_1$ in the inward direction)

In this case a simply-supported plate loaded at the inner radius is considered. The uniform radial load is in the outward direction. The curves for the following data are presented on Fig. 10.

$\nu_1 = \nu_2 = 1/3$

$t_1 = t_2 = 1.0$

$E_2/E_1 = 0$ (Hole), 0.05, 0.1, 0.15, 0.2, 0.3

$r_1/r_0 = 0.2, \ldots, 0.8$ and

$n = 0$ (Axisyummetric mode)

The curves on Fig. 10 correspond to a single annular plate with a flexible inclusion. The critical load for a single annular is very large for $r_1/r_0 < 0.2$, since the inplane stresses are very small in most part of the plate and are very large near the point of application. For $r_1/r_0 >$ 0.2 the stresses are distributed more uniformly within the entire plate.

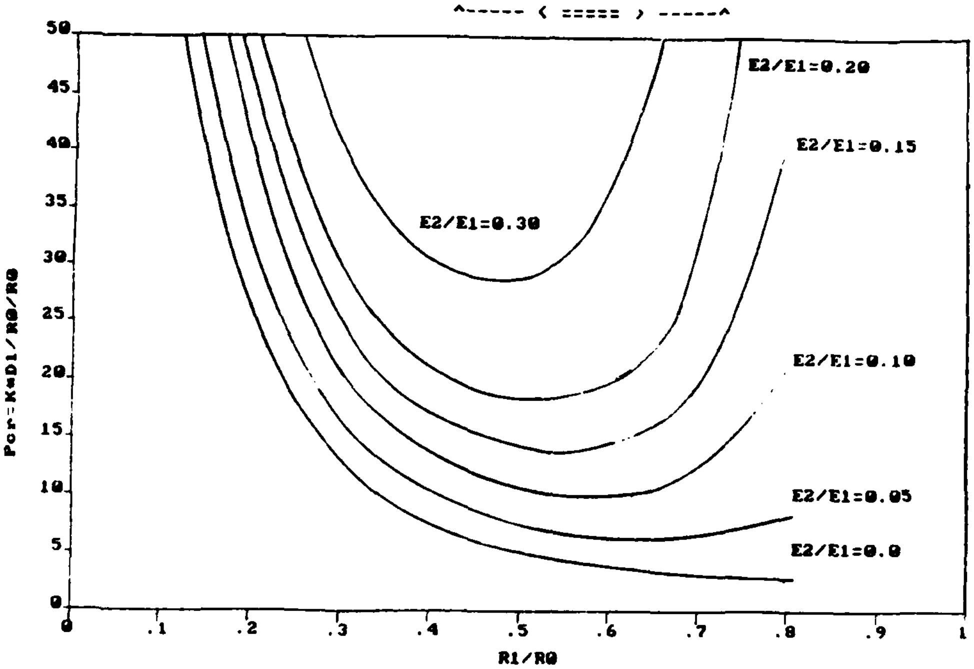


Fig. 10: P_{cr} versus r_1/r_o for various rigidity ratios with n = 0 and simply supported plate, loaded at the inner joint.

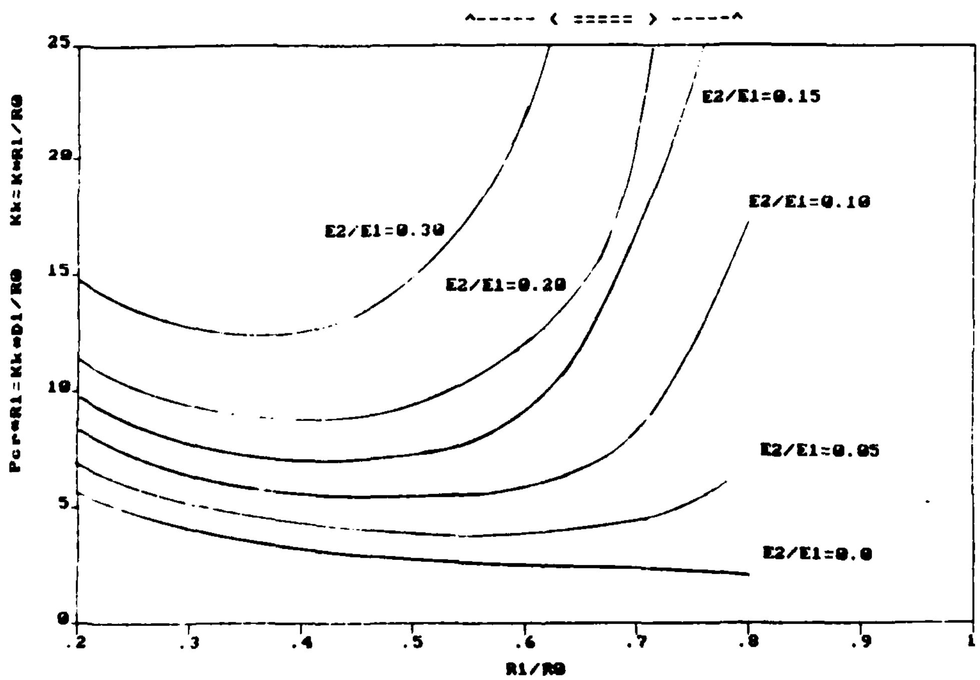


Fig. 11: The total buckling force ($P_{cr}r_1$) versus r_1/r_o for various rigidities, with n = 0 and simply-supported plate, loaded at the inner joint.

In the case of the plate with the flexible inclusion the buckling force becomes small around $r_1/r_0 \approx 0.55$. The critical force increase, for all values of E_2/E_1, as the radii ratio increase, since the tension in the central part is governing. The force for all r_1/r_0 increases with increasing of the rigidity ratios.

The "total" force, $p_{cr}\, r_1$, versus radii ratio r_1/r_0 is described on Fig. 11. In general, the trends of the "total" force appearing in the figure and the critical force described in the previous one are the same. The "total" buckling forces for $r_1/r_0 \leq 0.2$ are bounded and don't go to infinity as can be deducted from Fig. 10.

5. CONCLUSIONS

A buckling analysis of a multi-annular plate is described. The stiffness method is used to determine the primary state. The buckling equation is solved rigorously using the power series method. Convergence is proved either by the infinite series convergence test or by the accurate satisfaction of the differential equation condition. The versatility of the proposed solution is proved by the solved examples. The numerical examples reveal that the influence of the different annular parts and of the central part may or may not improve resistance to buckling. The influence of the various parameters, r_1/r_0, E_2/E_1, ν, n, is discussed to some extent for a two-part composite plate. A complete parametric study is recommended to understand the influence of the various parameters on the buckling behavior of multi-annular plates.

The proposed solution is also applicable to the buckling analysis of thin plates when resting on an elastic foundation, and even to the vibration analysis of multi-annular plates.

References

1. Dean, W. R., "The Elastic Stability of an Annular Plate, "Proceedings of the Royal Society of London, England, Ser. A., Vol. 106, 1924, pp. 268-285.

2. Willers, Fr. A., "Die Stabilitat von Krisringplatten," Zeitschrift fur angewandie Mathematik und Mechanik, Vol. 13, 1943, p. 253-254.

3. Federhofer, K. and Egger, H., "Knickung der auf Scherung beanspruchten kreisringplatte mit veranderlicher Dicke," Ingenieur-Archiv, Vol. 14, 1943, p. 155-166.

4. Federhofer, K., "Knicking oler kereisplatte und kreisring plate mit veranderlicher Dicke," Ingenieur-Archiv, Vol. 11, 1940, p. 224-240.

5. Olsson, R. G., "Uber Olie Knickung der Kreisring platten veranderlicher Dicke, "Ingenieur-Archiv, Vol. 12, 1941, p. 123-132.

6. Egger, H., "Knickung der Kreisplatte und Kreisring platte mit veranderlicher Dicke," Ingenieur-Archiv, Vol. 12, 1941, p. 190-200.

7. Olsson, R. G., "Uber axialsymmetrische knickung dunner kreisringpplatten," Ingenieur-Archiv, Vol. 8, 1937, p. 449-452.

8. Schubert, A., "*Die Beullast dunner Kreisringplatten die am Aussen und* Innenrand gleichmessigen Druck erfahren," Leitschrift fur angewandte Mathematic und Mechanik, Vol. 25/27, 1947, p. 123-124.

9. Yamaki, N., "Buckling of a Thin Annular Plate under Uniform Compression," Journal of Applied Mechanics, June 1958, pp. 267-273.

10. Meissner, E., "Uber das Knicken Kreisringformiger Scheiben," Schweizerche Bauzeitung, Vol. 101, 1933, p. 87-89.

11. Mansfield, E. H., "On the Buckling of Annular Plate", Quarterly J. of Mechanics and Applied Mechematics", Vol. 13, No. 1, Feb., 1960, pp. 16-23.

12. Balabueh, I. and Shapovalov, L. A., "Certain Problems of the Stability of Circular Plates in a Nonuniform Stress State," Insh. Zh. Mech. Tverd. Tele, No. 1, 1960, p. 60-69.

13. Bareeva, N. and Lizarev, A. D., "On the stability of an annular plate under the influence of radial tensile forces," Insh. Zh. Mech. Tverd. Tele, No. 2, 1966, p. 126-129.

14. Bareeva, N. and Lizarev, S. D., "Nonaxisymmetric modes of stability loss in annular plates in a nonuniform stress field", Insh. Zh. Mech. Tverd. Tele, No. 6, p. 103-105.

15. Lizarev, S. D. and Bareeva, G. N., "Stability of an elastically-fastened annular plate in a nonuniform stress field," Insh. Zh. Mech. Tverd. Tele, Vol. 5, No. 3, 1965, P. 483-491.

16. Lizarev, A. D. and Klenov, V.I., "Free Vibration of Cylindrically Anisotropic Annular Plates," (Translation), Priklashaya Mekhanika, Vol. 13 No. 7, 1977, pp. 76-81.

17. Lizarev, A. D., "Stability of Annular Plates in a Non-Uniform Stress Field", (Translation), Priklashaya Mekhanika, Vo. 16, No. 6, 1980, pp. 92-97.

18. Majumdar, S., "Buckling of a Thin Annular Plate Under Uniform Compression," AIAA Journal, No. 9, 1971, pp. 1701-1707.

19. Szillard, R., Theory and Analysis of Plates, Classical and Numerical Methods, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1974.

20. Markus, G., Theorie and Berechnung Rotations Symmetrischer Bauwerke, Werner-Verlag, Dusseldorf, 1967.

21. Rice, J. R., Numerical Methods, Software and Analysis; IMSL reference edition, McGraw-Hill, 1983.

22. Chia, C. Y., Nonlinear Analysis of Plates, New York, McGraw-Hill, 1980.

23. Gerald, C. F., Applied Numerical Analysis, Reading, Mass., Addison-Wesley series in Mathematics, Addison-Wesley, 1970.

Appendix A: Derivation of the Buckling Equation and Inplane Stiffness Coefficients.

A.1 Derivation of the equilibrium equations:

The nonlinear kinematic relations in polar coordinates, are [22]

$$\varepsilon_{rr} = u,_r + \frac{1}{2}(w,_r)^2$$

$$\varepsilon_{\theta\theta} = \frac{v,_\theta}{r} + \frac{u}{r} + \frac{1}{2}(w,_\theta)^2$$

$$\gamma_{r\theta} = v,_r + \frac{v}{r} - \frac{u,_\theta}{r} + \frac{w,_r w,_\theta}{r} \qquad \text{(A.1)}$$

$$\chi_{rr} = w,_{rr}$$

$$\chi_{\theta\theta} = \frac{w,_{\theta\theta}}{r^2} + \frac{w,_r}{r}$$

$$\chi_{r\theta} = \frac{w,_{r\theta}}{r} - \frac{w,_\theta}{r^2}$$

where: ε_{ij} = strains of the midplane $(i,j = r,\theta)$

u,v,w = displacement components of midplane points in the x,y, and z directions, respectively and

$(\ ),_i$ = differentiation with respect to i:

The equilibrium equations for isotropic material (see Fig. A.1) without body forces are:

$$(rN_{rr}),_r + N_{r\theta,\theta} - N_{\theta\theta} = 0$$

$$N_{\theta\theta,\theta} + (rN_{r\theta}),_r + N_{r\theta} = 0 \qquad \text{(A.2)}$$

$$D\nabla^4 w = \frac{1}{r}(rN_{rr}w,_r),_r + (N_{\theta\theta}\, w,_{\theta/r}),_\theta + (N_{r\theta}w,_r),_\theta + (N_{r\theta}w,_\theta),_r$$

where: $D = \dfrac{Et^3}{12(1-\nu^2)}$ = flexural rigidity of the plate.

N_{rr}, $N_{\theta\theta}$, $N_{r\theta}$ – Stress resultants (for sign convention see Fig. A.1)

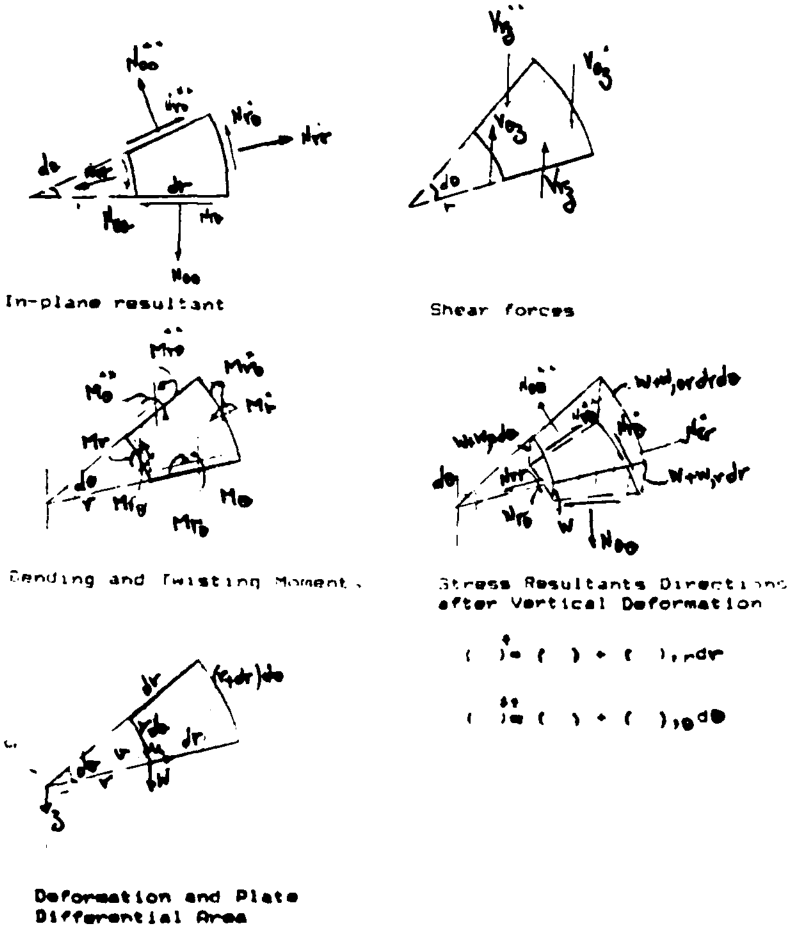


Fig. A.1 Stress Resultant, Moments and Shear forces and Deformations of a differential Plate Area

The Laplacian operator ∇^2, operating twice on w, yields

$$\nabla^4 w = w_{,rrrr} + \frac{2}{r} w_{,rrr} + \frac{1}{r^2} (- w_{,rr} + 2w_{,\theta\theta rr}) + \frac{1}{r^3} (w_{,r} - 2w_{,\theta\theta r})$$

$$+ \frac{1}{r^4} (4w_{,\theta\theta} + w_{,\theta\theta\theta\theta}) \qquad \text{(A.3)}$$

Next, the buckling equations are derived using the perturbation approach.

The stress resultants, and transverse displacement, in the buckled state, are given by:

$$N_{rr} = N^{o}_{rr} + N^{*}_{rr}$$

$$N_{\theta\theta} = N^{o}_{\theta\theta} + N^{*}_{\theta\theta} \qquad \text{(A.4)}$$

$$N_{r\theta} = N^{o}_{r\theta} + N^{*}_{r\theta}$$

$$w = w^{o} + w^{*}$$

where: $(\quad)^{o}$ = primary state parameters

$(\quad)^{*}$ = additional small parameters need to reach a buckled state.

Substituion of the quantities of Eqs. (A.4) into Eqs. (A.2) and some algebraic manipulations yield (note that, $|N^{*}_{ij}| << |N^{o}_{ij}|$ and $w^{o} = 0$)

$$(r N^{*}_{rr})_{,r} + N^{*}_{r\theta,\theta} - N^{*}_{\theta\theta} = 0$$

$$N^{*}_{\theta\theta,\theta} + (r N^{*}_{r\theta})_{,r} + N^{*}_{r\theta} = 0 \qquad \text{(A.5)}$$

$$D\nabla^4 w^{*} = \frac{1}{r} (r N^{o}_{rr} w^{*}_{,r})_{,r} + (N^{o}_{\theta\theta} w^{*}_{,\theta}/r)_{,\theta} + (N^{o}_{r\theta} w^{*}_{,\theta})_{,\theta} + (N^{o}_{r\theta} w^{*}_{,\theta})_{,r}$$

The expressions for the bending moments and transverse shear forces are [19]:

$$M_{rr} = -D\,[w,_{rr} + \nu\,(\frac{1}{r^2}\,w,_{\theta\theta} + \frac{1}{r}w,_r)]$$

$$M_{\theta\theta} = -D\,[w,_r/r + \frac{1}{r^2}\,w,_{\theta\theta} + \nu w,_{rr}]$$

$$M_{r\theta} = -(1-\nu)\,D\,[\frac{1}{r}\,w,_{r\theta} - \frac{1}{r^2}\,w,_\theta]$$

$$V_{rz} = \frac{1}{r}\,[(M_{rr}r),_r + M_{r\theta,\theta} - M_{\theta\theta}] = -D\,(\nabla^2 w),_r \tag{A.6}$$

$$V_{\theta z} = \frac{1}{r}\,[M,_{\theta\theta} + M_{r\theta,r}\,r + 2M_{r\theta}] = -D\,\frac{1}{r}(\nabla^2 w),_\theta$$

and the edge effective shear forces are:

$$Q_r = V_{rz} + \frac{1}{r}\,M_{r\theta,\theta}$$

$$Q_\theta = V_{\theta z} + M_{r\theta,r}$$

A.2: Primary state analysis

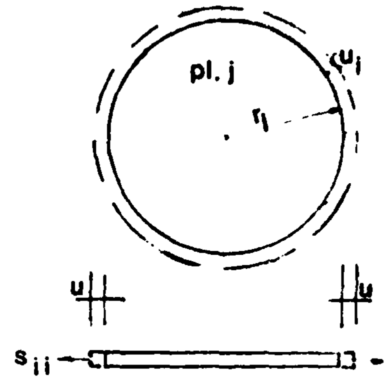
The stresses and the deformations at the adjacent joints of the annular parts, are calculated by using the stiffness approach. The formulae for stresses are taken from Ref. [20].

A.2.1 Stiffness Coefficient $[^{S}\ell j]$

The stiffness coefficients are determined separately for the circular and annular parts.

A.2.1.1 Circular plate stiffness

$$s_{ii} = \frac{E_j}{r_i(1 - \nu_{i+1})}$$



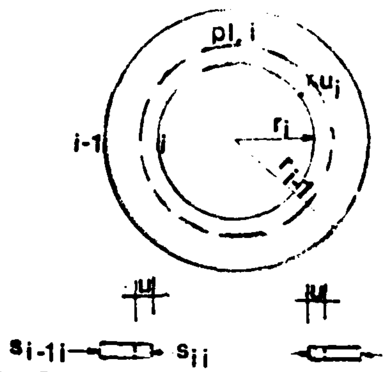
and the stresses and deformation are:

$$N^{o}_{rr} = s_{ii} = u_i = N^{o}_{\theta\theta}$$

$$u(r) = \frac{r}{r_i} u_i$$

A.2.1.2 Annular plate stiffness [see [20])

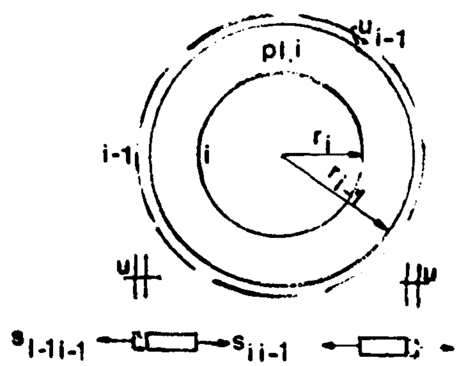
(a) <u>Deformation, u_i, at inner radius</u>



$$s_{ii} = \frac{E_i[(1+\nu_i) + \beta_i^2(1-\nu)]}{r_i}$$

$$s_{i-1i} = \frac{2E_i}{r_i\ (\beta_i^2-1)\ (1-\nu_i)}$$

(b) <u>Deformation, u_{i-1} at outer radius (see [20])</u>



$$s_{i-1i-1} = \frac{E_i\beta_i[1+\nu\ ) + \dfrac{1-\nu_i}{\beta_i^2}}{(\beta_i^2-1)\ \ (1-\nu_i^2)}$$

$$s_{ii-1} = \frac{2E_i\beta_i}{r_i(\beta_i^2-1)\ (1-\nu_i^2)}$$

The relation between the off-diagonal terms is:

$$r_i \, s_{i-1\, i} = r_{i-1} \, s_{i\, i-1}$$

and the stressses and deformation in the plate are:

$$(r_i \le r \le r_{i-1})$$

$$N^{o}_{rr}(r) = \frac{E_i}{r_i(\beta_1^2-1)(1-\nu_i^2)} \left[(1+\nu_i)(-u_i+\beta_i u_{i-1}) + \frac{\beta_i(1-\nu_i)r_i^2}{r^2}(-\beta_i u_i + u_{i-1})\right]$$

$$N^{o}_{\theta\theta}(r) = \frac{E_i}{r_i(\beta_i^2-1)(1-\nu_i^2)} \left[(1+\nu_i)(-u_i+\beta_i u_{i-1}) - \frac{\beta_i(1-\nu_i)r_i^2}{r^2}(-\beta_i u_i + u_{i-1})\right]$$

$$u(r) = \left[u_{i-1}\,\beta_i\left(1-\left(\frac{r_i}{r}\right)^2\right) + u_i\left(\frac{r_{i-1}^2}{r^2}-1\right)\right]\frac{r}{r_i\,(\beta_i^2-1)}$$

Appendix B: Various Solutions of the Buckling Equation

B.1 Solution of the buckling equation*

The buckling equation, after using separation of variables reads:

$$r^4 F^{IV} + 2r^3 F^{III} - r^2(1 + 2n^2 + \beta_{ci}^4 \alpha_i^2 - r^2 \alpha_i^2) F^{II} + r(1+2n^2 + \beta_{ci} \alpha_i^2 + r^2 \alpha_i^2) F^I$$

$$- F [n^2(4 - n^2 + \beta_{ci} \alpha_i^2 + r^2 \alpha_i^2) = 0 \qquad \text{(B.1)}$$

where $\alpha_i^2 = \dfrac{N_{o_i}}{D_i}$ and $\beta_{ci} = \dfrac{N_{o_{ci}}}{N_{o_i}}$

We first assume a power series type of solution

$$F_i(r) = \sum_{j=0}^{\infty} A_j r^{j+s} \qquad \text{(B.2)}$$

Substitution of this solution into Eq. (B.1) (after some algebric manipulations) yields

$$\sum_{j=0}^{\infty} A_j r^{j+s} \{ (j + s - 1)^4 - (2 + 2n^2 + \beta_{ci} \alpha_i^2) (j + s - 1)^2 +$$

$$+ [(n-1)^2 + \beta_{ci} \alpha_i^2 (1 - n^2)] \} + \sum_{j=0}^{\infty} A_j r^{j+s+2} \alpha_i^2 [(j + s)^2 - n^2] = 0 \qquad \text{(B.3)}$$

---

*For simplicity the subscripts, ni or i or k are omitted, throughout this appendix.

The indicial equation reads (j=0):

$$(s-1)^4 - (2 + 2n^2 + \beta_{ci}\alpha_i^2)\,(s-1)^2 + [(n-1)^2 + \beta_{ci}\alpha_i^2\,(1-n^2)] = 0 \qquad \text{(B.4)}$$

and the roots of the quartic equation are:

$$(s-1)^2_{1,2} = \left[\left(1+n^2 + \frac{\beta_{ci}\alpha_i^2}{2}\right)\right] + \sqrt{4n^2 + 2n^2\beta_{ci}\alpha_i^2 + \frac{\beta_{ci}^2\alpha_i^4}{4}} = \gamma + \delta^{1/2}$$

$$(s-1)^2_{3,4} = \left[\left(1+n^2 + \frac{\beta_{ci}\alpha_i^2}{2}\right)\right] - \sqrt{4n^2 + 2n^2\beta_{ci}\alpha_i^2 + \frac{\beta_{ci}^2\alpha_i^4}{4}} = \gamma - \delta^{1/2}\,, \qquad \text{(B.5)}$$

thus

$$\begin{aligned} s_{1,2} &= 1 \pm \sqrt{\gamma + \delta^{1/2}} \\ s_{3,4} &= 1 \pm \sqrt{\gamma - \delta^{1/2}} \end{aligned} \qquad \text{(B.6)}$$

where γ is equal to the terms appearing in first brackets and δ equals to the terms appearing under the square root sign in Eq. (B.5). The recurrence formulae is:

$$A_j = -\frac{\alpha_i^2[(j+s-2)^2 - n^2]}{[(j+s-1)^2 - \gamma]^2 - \delta}\,A_{j-2} = \frac{\alpha_i^2[(j+s-2)^2 - n^2]\quad A_{j-2}}{[(j+s-1)^2 - (\gamma+\delta^{1/2})]\,[(j+s+1)^2 - (\gamma-\delta^{1/2})]}$$

where γ and $\delta^{1/2}$ are the expressions appearing in Eq. (B.5).

The solution, in general, is of the form

$$F_k = \sum_{j=0}^{\infty} A_j\, r^{j+s_k} \qquad \text{(B.8)}$$

(k = 1 to 4)

(j = 0,2, ... even)

B.2 Solutions of the Differential Equation

B.2.1: $\beta_{c_i} > 0$

The solutions for a positive β_{c_i} are derived next.

B.2.1.1. n=0

For this case the roots are:

$$\begin{aligned} s_1 &= 1+(1+\beta_{c_i}\alpha_i^2)^{1/2} = 1+\phi_i \\ s_2 &= 1-(1+\beta_{c_i}\alpha_i^2)^{1/2} = 1-\phi_i \\ s_3 &= 2 \\ s_4 &= 0 \end{aligned} \tag{B.9}$$

where $\gamma+\delta^{1/2} = \phi_i^2 \; : \; \gamma-\delta^{1/2} = 1 \qquad \gamma = 1+\beta_{c_i}\alpha_i^2/2 \; : \; \delta = \beta_{c_i}\alpha_i^2/4 :$

The recurrence formula is:

$$A_j = -\frac{(j+s-2)^2\alpha_i^2 A_{j-2}}{[(j+s-1)^2-\phi_i^2][(j+s-1)^2-1]} = \frac{\alpha_i^2 (j+s-2)^2 A_{j-2}}{[(j+s-1)^2-\phi_i^2][(j+s)^2-2(j+s)+1-1]}$$

$$= -\frac{\alpha_i^2 (j+s-2) A_{j-2}}{[(j+s-1)^2-\phi_i^2](j+s)} \tag{B.10}$$

The general solution is

$$F = A_0 r^s \left\{ 1 - \frac{\alpha_i^2 r^2 s}{[(s+1)^2-\phi_i^2](2+s)} + \cdots + \frac{(-1)^{N/2}\alpha_i^N r^N \prod_{j=2}^{N}(j+s-2)}{\prod_{j=2}^{N}[(j+s-1)^2-\phi_i^2](j+s)} \right\} \tag{B.11}$$

The solution for the various roots are:

$$F_1(s=s_2=1-\phi_i) = A_0 r^{1-\phi_i} \left\{ 1 - \frac{\alpha_i^2 r^2 (1-\phi_i)}{[(2-\phi_i)^2-\phi_i^2](3-\phi_i)} + \cdots \cdots + \frac{(-1)^{N/2}\alpha_i^N r^N \prod_{j=2}^{N}(j+1-\phi_i)}{\prod_{j=2}^{N}[(j-\phi_i)^2-\phi_i^2](j+1-\phi_i)} \right\}$$

$$F_2(s=s_1=1+\phi_i)=A_0 r^{1+\phi_i}\left\{1-\frac{\alpha_i^2 r^2(1+\phi_i)}{[(2-\phi_i)^2-\phi_i^2](3+\phi_i)}+\cdots\cdots\right.$$

$$\left.\cdots+\frac{(-1)^{N/2}\alpha_i^N r^N \prod_{j=2}^{N}(j+\phi_i+1)}{\prod_{j=2}^{N}[(j+\phi_i)^2-\phi_i^2](j+1+\phi_i)}\right\}$$

$$F_3=r^2\left\{\frac{1}{2}-\frac{\alpha_i^2 r^2}{4(3^2-\phi_i^2)}+\cdots\cdots+\frac{(-1)^{N/2}\alpha_i^N r^N}{(N+2)\prod_{j=2}^{N}[(j+1)^2-\phi_i^2]}\right\}$$

$$F_4=C\quad(A_2=A_4=\ldots\ldots=A_N=0)\qquad(B.12)$$

$$F(r)=C_1F_1+C_2F_2+C_3F_3+C_4F_4\quad(\phi_i=Re(\phi_i),\ \beta_{ci}>0)\qquad(B.13)$$

B.2.1.2: n=1

The roots in this case become:

$$s_1=1+\sqrt{4+\beta_{ci}\alpha_i^2}=1+\psi_i$$
$$s_2=1-\sqrt{4+\beta_{ci}\alpha_i^2}=1-\psi_i\qquad(B.14)$$
$$s_3=1$$
$$s_4=1$$

where: $\gamma=2+\beta_{ci}\alpha_i^2/2\ :\ \delta=(2+\beta_{ci}\alpha_i^2/2)^2$

The recurrence formula is:

$$A_j=-\frac{[(j+s-2)^2-1]\alpha_i^2A_{j-2}}{[(j+s-1)^2-\psi_i^2](j+s-1)^2}=-\frac{(j+s-3)\alpha_i^2A_{j-2}}{[(j+s-1)^2-\psi_i^2](j+s-1)}\qquad(B.15)$$

and the various coefficients are:

$$A_0 = \frac{1}{s-1} \; : \; A_2 = -\frac{\alpha_i^2}{[(s+1)^2-\psi_i^2](s+1)} \; : \; A_N = \frac{(-1)^{N/2}\alpha_i^N}{(s+N-1)\prod_{j=2}^{N}[(j+s-1)^2-\psi_i^2]}$$

The general solution reads as follows

$$F = r^s\left\{\frac{1}{s-1} - \frac{\alpha_i^2 r^2}{(s+1)[(s+1)^2-\psi_i^2]} + \cdots + \frac{(-1)^{N/2}\alpha_i^N r^N}{(s+N-1)\prod_{j=2}^{N}[(j+s-1)^2-\psi_i^2]}\right\} \quad (B.16)$$

The solution for the various roots are:

$$s_1 = 1+\psi_i \; : \; F_1 = r^{1+\psi_i}\left\{\frac{1}{\psi_i} - \frac{\alpha_i^2 r^2}{[(2+\psi_i)^2-\psi_i^2](2+\psi_i)} \cdots + \frac{(-1)^{N/2}\alpha_i^N r^N}{(N+\psi_i)\prod_{j=2}[(j+\psi_i)^2-\psi_i^2]}\right.$$

$$s_2 = 1-\psi_i \; : \; F_2 = r^{1-\psi_i}\left\{-\frac{1}{\psi_i} - \frac{\alpha_i^2 r^2}{[(2-\psi_i)^2-\psi_i^2](2-\psi_i)} + \cdots + \frac{(-1)^{N/2}\alpha_i^N r^N}{(N-\psi_i)\prod_{j=2}^{N}[(j-\psi_i)^2-\psi_i^2]}\right.$$

$$s_3 = 1 : \; F_3 = r \quad (A_2 = A_4 = \dots A_N = 0) \qquad (B.17)$$

Since $s_4 = s_3$ the following solution is assumed (see [23]):

$$F_4 = C_1 r \ln(r) + \sum_{j=0}^{\infty} b_j r^{j+s_4} \qquad (B.18)$$

Substitution of the solution in the D.E. (see eq. B.1) yields:

$$D.E.(F_3\ln(r)) = 4F_3''' r^3 - (4n^2+2\beta_{ci})(F_3' r - F_3) + \alpha_i^2 r^2 (2F' r) =$$

$$= -(4n^2+2\beta_{ci})(1\cdot r - r) + \alpha_i^2 r^2 (2r) = 2\alpha_i^2 r^3$$

$$D.E.\left(\sum b_j r^{j+s_4}\right) = r^{s_4}\sum_{j=2}^{N} r^j\Big\{\left[(j+s_4-1)^2-\psi_i^2\right](j+s_4-1)^2 b_j + \alpha_i^2\left[(j+s_4-2)^2-n^2\right]b_{j-2}\Big\} + \left[(s-1)^2-\psi_i^2\right](s-1)^2 b_0$$

where D.E(F$_K$) = Eq. (B.1) with F=F$_K$. Thus, the result is:

$$D.E.(F_4) = (s_4-1)^2\left[(s_4-1)^2-\psi_i^2\right]b_0 r + r\sum_{j=2}^{N} r^j\left[(j+s_4-1)^2-\psi_i^2\right](j+s_4-1)^2 + \alpha_i^2\left[(j+s_4-2)^2-n^2\right]b_{j-2} + 2\alpha_i^2 r^3 \qquad (B.19)$$

Equating the coefficients of the various powers of r to zero yields:

$$r^{s_4} = r:(j=0):(s_4-1)^2\left[(s_4-1)^2-\psi_i^2\right]b_0 = 0 \Rightarrow b_0 = 0$$

$$r^{s_4+2} = r^3:(j=2):\left[(2+s_4-1)^2-\psi_i^2\right](2+s_4-1)^2 b_2 + \alpha_i^2\left[(2+s_4-2)^2-1\right]b_0 + 2\alpha_i^2 = 0 \rightarrow b_2 = \frac{2\alpha_i^2}{\left[(s_4+1)^2-\psi_i^2\right](s_4+1)^2}$$

$$\vdots$$

$$r^{j+s_4}:(j=j):\left[(j+s_4-1)^2-\psi_i^2\right](j+s_4-1)^2 b_j + b_{j-2}\alpha_i^2\left[(j+s_4-2)^2-n^2\right] = 0$$

$$b_j = \frac{-\alpha_i^2\left[(j+s_4-2)^2-n^2\right]}{\left[(j+s_4-1)^2-\psi_i^2\right](j+s_4-1)} b_{j-2} \qquad (B.20)$$

hence the fourth solution is:

$$F_4 = r\ln(r) + \sum_{j=2}^{N} b_j r^{j+s_4} \qquad (B.21)$$

B.2.1.3: n>=2

When n>=2 the roots are:

$$
\begin{aligned}
s_1 &= 1+\sqrt{\mu+\delta^{1/2}} = 1+\psi_1 \\
s_2 &= 1-\sqrt{\mu+\delta^{1/2}} = 1-\psi_1 \\
s_3 &= 1+i\sqrt{-\mu+\delta^{1/2}} = 1+i\psi_2 \\
s_4 &= 1-i\sqrt{-\mu+\delta^{1/2}} = 1-i\psi_2
\end{aligned}
\qquad (B.22)
$$

where:

$$
\begin{aligned}
\mu &= 1+n^2+\beta_{ci}\alpha_i^2/2 \\
\delta &= (2n^2+\beta_{ci}\alpha_i^2/2)^2 - 4n^2(n^2-1) \qquad (B.22.1)\\
\psi_1^2 &= \mu+\delta^{1/2} \\
-\psi_2^2 &= \mu-\delta^{1/2}
\end{aligned}
$$

The recurrence formula is:

$$
A_j = -\frac{\alpha_i^2\left[(j+s_4-2)^2-n^2\right]}{\left[(j+s_4-1)^2-\psi_1^2\right]\left[(j+s_4-1)^2+\psi_2^2\right]}A_{j-2} \qquad (B.23)
$$

Note that the detailed solution is described in article B.2.1.4.

B.3.1.4: A general solution for a case with complex roots

The complex roots are described by:

$$
S(I,k) = SR(I,k) + i\,SI(I,k) \qquad (B.24)
$$

and the recurrence formula in general is:

$$
A_j = -\frac{\left[(j+s-2)^2-n^2\right]\alpha_i^j}{\text{Denom}}A_{j-2} \qquad (B.25)
$$

where:

$$
\begin{aligned}
\text{Denom} = {} & (j+s)(j+s-1)(j+s-2)(j+s-3) + 2(j+s-1)(j+s-2)(j+s) - \\
& + (1+2n^2+\beta_{ci}^2)(j+s)(j+s-1) + (1+2n^2+\beta_{ci}^2)(j+s) - \\
& + n^2(4-n^2+\beta_{ci}^2)
\end{aligned}
\qquad (B.25.1)
$$

After some algebraic manipulations we obtain:

$$Denom = (j+s-1)^4 - 2\mu\,(j+s-1)^2 + (1-n^2)(1-n^2+\beta c_i^2) = DenomR + i\,DenomI \qquad (B.26)$$

where:

$$\begin{aligned} T1R1 &= [(j+SR-1)^2 - SI^2] \\ T1I1 &= 2[(j+SR-1)\,SI] \\ T1R2 &= T1R1^2 - T1I1^2 \\ T1I2 &= 2\cdot T1R1\cdot T1R2 \\ DenomR &= T1R2 - 2\mu\,T1R1 + (1-n^2)(1-n^2+\beta c_i^2) \\ DenomI &= T1I2 - 2\mu\,T1I1 \end{aligned} \qquad (B.26.1)$$

The nominator is:

$$Anom = (j+s-2)^2 - n^2 \quad = AnomR + i\,AnomI \qquad (B.27)$$

where:

$$\begin{aligned} AnomR &= (j+SR-2)^2 - SI^2 - n^2 \\ AnomI &= 2(j+SR-2)(SI) \end{aligned} \qquad (B.27.1)$$

The general recurrence formula, using complex notation, becomes:

$$A_j = (A_{jR} + i\,A_{jI})\,\alpha_i^2\,A_{j-2} \qquad (B.28)$$

where:

$$\begin{aligned} A_{jR} &= -\frac{AnomR\cdot DenomR + AnomI\cdot DenomI}{DenomR^2 + DenomI^2} \\ A_{jI} &= -\frac{AnomI\cdot DenomR - AnomR\cdot DenomI}{DenomR^2 + DenomI^2} \end{aligned} \qquad (B.28.1)$$

For $s_k = s_{k-1}$, it can be easily shown that the recurrence equation is:

$$B_j = (A_{jR} - iA_{jI})\, \alpha_i^2\, A^*_{j-2} \qquad (B.29)$$

Hence the general solution is:

$$\bar{F}_1 = \bar{A}_o r^{sR} e^{isI(\ln r)} \left[1 - (A_{2R} + iA_{2I})\alpha_i^2 r^2 + \cdots + (-1)^{N/2} (\alpha_i r)^N \prod_{j=2}^{N} (A_{jR} + iA_{jI})\right]$$

$$\bar{F}_2 = \bar{B}_o r^{sR} e^{-isI(\ln r)} \left[1 - (A_{2R} - iA_{2I})\alpha_i^2 r^2 + \cdots + (-1)^{N/2} (\alpha_i r)^N \prod_{j=2}^{N} (A_{jR} - iA_{jI})\right]$$

$$(B.30)$$

If $F_1 = \bar{F}_1 + \bar{F}_2$ and $F_1 = \bar{F}_1 - \bar{F}_2$ with $\bar{A}_o = \bar{B}_o = 1/2$ then, after some algebraic manipulations we get

$$F_1 = r^{SR} \left[S_1 \cos(sI \ln(r)) - S_2 \sin(sI \ln(r))\right]$$

$$F_2 = r^{SR} \left[S_1 \sin(sI \ln(r)) - S_2 \cos(sI \ln(r))\right] \qquad (B.31)$$

where:

$$S_1 = 1 - A_{2R}\alpha_i^2 r^2 + \cdots + (-1)^{N/2} (\alpha_i r)^N A_{NR}$$

$$S_2 = -A_{2I}\alpha_i^2 r^2 + \cdots + (-1)^{N/2} (\alpha_i r)^N A_{NI} \qquad (B.31.1)$$

B.2.2: $\beta_{e1}=0$

The various solutions for $\beta_{e1}=0$ are determined next.

B.2.2.1 n=0

The roots are: $S_1 = S_2 = 2$ (B.32)

$S_3 = S_4 = 0$

with $\gamma=1$, $\delta=0$. The recurrence formula is (for $s\neq 0$):

$$A_j = -\frac{(j+s-2)^2 \alpha_i^2 A_{j-2}}{[(j+s-1)^2-1]^2} = -\frac{\alpha_i^2}{(j+s)^2} A_{j-2} \qquad (B.33)$$

Hence, the general solution is:

$$F_1 = 1 - \frac{\alpha_i^2 r^2}{2^2} + \frac{\alpha_i^4 r^4}{4^2 2^2} + \cdots + \frac{(-1)^{N/2}(\alpha_i r)^{N+2}}{\prod_{j=0}^{N}(j+2)^2} = J_0(\alpha_i r)$$

$$F_2 = \left.\frac{\partial F_1}{\partial s}\right|_{s=s_1} = Y_0(\alpha_i r)$$

(B.34)

$$F_3 = 1 \quad (A_2 = A_4 = \cdots A_N = 0)$$

$$F_4 = \ln(r) \quad (A_2 = A_4 = \cdots = A_N = 0)$$

where J (,r) and Y (,r) are Bessel functions of the first and second kind, respectively of zero order.

B.2.2.2: n=0

The roots are: $s_1 = 2+n$

$s_2 = 2-n$ (B.35)

$s_3 = -n$

$s_4 = n$

and $\gamma = 1 + n^2 \; ; \; \Gamma = 4n^2$

It can be shown, very easily, that the various solutions are:

$$
\begin{aligned}
F_1 &= J_n(\alpha_i r) \\
F_2 &= Y_n(\alpha_i r) \qquad\qquad (B.36) \\
F_3 &= r^{-n} \\
F_4 &= r^{n}
\end{aligned}
$$

where $J_n(\alpha_i r)$ and $Y_n(\alpha_i r)$ are Bessel functions of the first and second kind respectively of order n.

B.2.3. $\beta_{c1}<0$

The solutions for n=0 and n>=1 are determined in the next chapters.

B.2.3.1: n=0

The roots are:

$$
\begin{aligned}
s_1 &= 2 \\
s_2 &= 0 \\
s_3 &= 1+(1-|\beta_{ci}|\alpha_i^2)^{1/2} = 1+\phi_i \\
s_4 &= 1-(1-|\beta_{ci}|\alpha_i^2)^{1/2} = 1-\phi_i
\end{aligned}
\qquad (B.37)
$$

where $\gamma = 1-|\beta_{ci}|\alpha_i^2/2 \;:\; \delta^{1/2} = \beta_{ci}\alpha_i^2/2 \;:\; \gamma+\delta^{1/2}=1 \;:\; \gamma-\delta^{1/2} = 1-|\beta_{ci}|\alpha_i^2$

For $abs(\beta_{c1})\alpha_1^2=1$ the roots are:

$$
\begin{aligned}
s_1 &= 2 \\
s_2 &= 0 \\
s_3 &= s_4 = 1
\end{aligned}
\qquad (B.38)
$$

The first two solutions appear in art. B.3.1.1. The last two are derived next.
The recurrence formula is:

$$A_j = \frac{-\alpha_i^2(j+s_3-2)^2}{[(j+s_3-1)^2][(j+s_3-1)^2-1]}A_{j-2} = -\frac{\alpha_i^2(j+s_3-2)}{(j+s_3-1)^2(j+s)}A_{j-2} \qquad (B.39)$$

Hence for $s_3=1$ we have:

$$A_j = -\frac{\alpha_i^2(j-1)}{j(j+1)}A_{j-2} \qquad (B.40)$$

The third solution then is:

$$F_3 = r\left[1-\frac{\alpha_i^2 r^2}{2\cdot 3}+\ldots\ldots+\frac{(-1)^{N/2}(\alpha_i r)^N \prod_{j=2}^{N}(j-1)}{\prod_{j=2} j(j+1)}\right]A_0 \qquad (B.41)$$

The fourth solution is assumed to have the following form:

$$F_4 = C_1 F_3 \ln(r) + \sum_{j=0}^{\infty} b_j r^{j+s_4} \qquad (B.42)$$

Substitution of the solution into the D.E Eq. (B.1), yields:

$$D.E.\left(C_1 F_3 \ln(r)\right) = C_1\Big\{\sum_{j=2}^{N}\Big[\big[(j+s_3-1)-2\beta_{c_i}^2(j+s_3-1)\big]\big[4(j+s_3)(j+s_3-2)$$

$$-4n^2\big]A_j r^j + 2\alpha_i^2(j+s_3-2)A_{j-2}r^j\Big\} +$$

$$+\big[s_3(s_3-1)(s_3-2)-2\beta_{c_i}^2-4n^2(s_3-1)\big]\cdot A_0\Big\}$$

$$D.E.\left(\sum b_j r^{j+s_4}\right) = r\sum_{j=2}^{N} r^j\left\{b_j\,Den + \alpha_i^2\,Anom\,b_{j-2}\right\} +$$

$$(s_4-1)^2(s_4-2)s\cdot r\,b_0 \qquad (B.43)$$

where:

$$Den = (j+s_4-1)^2(j+s_4-2)(j+s_4)$$

$$Anom = (j+s-2)^2 - n^2$$

Equating the coefficients of the various powers of r to zero yields:

$$r:(j=0):\; s_3(s_3^{\overset{0}{}}-1)(s_3-2)A_0C_1 + (s_4-1)^{\overset{0}{}2}(s_4-2)\cdot s\cdot b_0 = 0$$

we choose : $C_1 = 1.0\,;\; b_0 = 0$

$$r^3:(j=2):\;(s_4+1)^2 s_4(2+s_4)b_2 + \alpha_i^2 s_4 b_0 + \big\{4(s_3+1)\big[(2+s_3)s_3\big]A_2 +$$

$$+2\alpha_i^2 s_3 A_0 = 0 \implies b_2 = -\frac{DA1}{(s_4+1)^2 s_4(2+s_4)} \qquad (B.44)$$

where DA1 equals the terms in the large brackets of Eq. (B.44). The recurrence formula for b_j is:

$$b_j = \frac{-DA1 - b_{j-2}\,\alpha_i^2\,Anom}{Den} \qquad (B.45)$$

B.2.3.2: n=1

The roots are:
$$S_1 = 1 + (4 - |\beta_{ci}|\alpha_i^2)^{1/2} = 1 + \phi_i$$
$$S_2 = 1 - (4 - |\beta_{ci}|\alpha_i^2)^{1/2} = 1 - \phi_i \qquad \text{(B.46)}$$
$$S_3 = 1$$
$$S_4 = 1$$

where $\mu = 2 - |\beta_{ci}|\alpha_i^2/2 \,:\, \delta = (2 - |\beta_{ci}|\alpha_i^2/2)^2 \,:\, \mu + \delta^{1/2} = \phi_i^2 \quad \gamma - \delta^{1/2} = 0$

The third and fourth solutions are the same as those derived in art B.2.1.2.
The first two roots depends on $\emptyset_i$. If abs(β_{ci}) $_i$<4 then those two roots are real and the solutions are the same as those derived for positive β_{ci} $_i$ (see art. B.2.1.2).
For abs(β_{ci}) $_i$=4 the first two roots are equal to 1, namely all four roots are equal. The four solutions are:

$$F_1 = r$$
$$F_2 = r\ln(r) + \sum_{j=0}^{\infty} b_j r^{j+1}$$
$$F_3 = r(\ln(r))^2 + \sum_{j=0}^{\infty} c_j r^{j+1} \qquad \text{(B.47)}$$
$$F_4 = r(\ln(r))^3 + \sum_{j=0}^{\infty} d_j r^{j+1}$$

The various coefficients b_j, c_j, d_j, are determined by the same procedure appearing in the previous article.
For abs(β_{ci})α_i^2>4, the first two roots are complex and equal to:

$$S_1 = 1 + i\phi_i$$
$$S_2 = 1 - i\phi_i \qquad \text{(B.48)}$$

The general solution for this type of roots appears in art. B.2.1.4.

B.2.2.3: n>=2

The roots are:
$$S_1 = 1 + \sqrt{\mu + \delta^{1/2}} = 1 + \psi_1$$
$$S_2 = 1 - \sqrt{\mu + \delta^{1/2}} = 1 - \psi_1$$
$$S_3 = 1 + \sqrt{\mu - \delta^{1/2}} = 1 + \psi_2 \qquad \text{(B.49)}$$
$$S_4 = 1 - \sqrt{\mu - \delta^{1/2}} = 1 - \psi_2$$

For $abs(\beta_{ci})\alpha_i^2 < 2.15$ (see Table 1), $\delta^{1/2} > 0$, ψ_1 is a real number and ψ_2 is complex. Hence the roots are:

$$
\begin{aligned}
s_1 &= 1+\psi_1 \\
s_2 &= 1-\psi_1 \\
s_3 &= 1+i\psi_2 \\
s_4 &= 1-i\psi_2 .
\end{aligned}
\qquad (B.50)
$$

where:

$$
\begin{aligned}
\psi_1^2 &= \mu+\delta^{1/2} \\
-\psi_2^2 &= \mu-\delta^{1/2} \\
\mu &= 1+n^2-|\beta_{ci}|\alpha_i^2/2 \\
\delta &= (2n^2-|\beta_{ci}|\alpha_i^2/2)^2-4n^2(n^2-1)
\end{aligned}
\qquad (B.50.1)
$$

The solutions for these type of roots are given in art. B.2.1.3. That solution also covers the range $0 < abs(\beta_{ci})_1 < 2$.
If $abs(\beta_{ci})\alpha_1^2 >$ then $\delta^{1/2}$ is negative meaning the roots are:

$$
\begin{aligned}
s_1 &= 1+\sqrt{\mu+i\delta^{1/2}} = 1+(\psi_R+i\psi_I) \\
s_2 &= 1-\sqrt{\mu+i\delta^{1/2}} = 1-(\psi_R+i\psi_I) \\
s_3 &= 1+\sqrt{\mu-i\delta^{1/2}} = 1+(\psi_R-i\psi_I) \\
s_4 &= 1-\sqrt{\mu-i\delta^{1/2}} = 1\ \ (\psi_R-i\psi_I)
\end{aligned}
\qquad (B.51)
$$

where:

$$
\begin{aligned}
\psi_R &= R^{1/2}\cos\vartheta/2 &&: \quad \mu = R\cos\vartheta \\
\psi_I &= R^{1/2}\sin\vartheta/2 &&: \quad \delta^{1/2} = R\sin\vartheta \\
R &= (\mu^2+\delta)^{1/2} &&: \quad \operatorname{arctg}\vartheta = \delta^{1/2}/\mu
\end{aligned}
\qquad (B.51.1)
$$

and $s_1=s_3^*$ and $s_2=s_4$ (The $(\ )^*$ means a complex conjugate number).
The general solution for this type of roots appears in art. B.2.1.4

B.2.4. $\alpha_i^2=0, \beta_{ci}\neq 0$

In this case the uniform part of the stress field is zero and the differential equation (Eq. B.1), becomes an Euler type. The solution for this type of equation is given by functions then by power series.
The indicial equation is the same as for the power series solution, Eq. (B.4). The various roots are described in Table 1. The solutions for the various types of roots is derived next.
The roots in general are:

$$s_{1,2}=1\pm\sqrt{\gamma+\delta^{1/2}} \qquad s_{3,4}=1\pm\sqrt{\gamma-\delta^{1/2}} \qquad (B.52)$$

where:

$$\gamma=1+n^2+\beta_{ci}\alpha_i^2/2 \qquad \delta=4n^2+2n^2\beta_{ci}\alpha_i^2+\beta_{ci}^2\alpha_i^4/4 \qquad (B.52.1)$$

B.2.4.1: $n=0$, $\beta_{ci}>0$

The D.E. is equal to:

$$r^4F^{IV}+2r^3F'''-r^2(1+\beta_{ci}\alpha_i^2)F''+r(1+\beta_{ci}\alpha_i^2)F'=0 \qquad (B.53)$$

and the roots are:

$$s_1=1+\sqrt{1+\beta_{ci}\alpha_i^2}=1+\phi_i \qquad s_2=1-\sqrt{1+\beta_{ci}\alpha_i^2}=1-\phi_i \qquad s_3=2 \qquad s_4=0 \qquad (B.54)$$

The general solution is:

$$F=C_1r^{1+\phi_i}+C_2r^{1-\phi_i}+C_3r^2+C_4 \qquad (B.55)$$

B.2.4.2: $n=0$, $\beta_{ci}<0$

The roots are:

$$s_{1,2}=1\pm i\phi_i \qquad s_3=2; \quad s_4=0 \qquad (B.56)$$

Hence the solution is:

$$F = C_1 r^{(1+i\phi_i)} + C_2 r^{(1-i\phi_i)} + C_3 r^2 + C_4 \qquad (B.57)$$

In terms of trigonometric functions, the solution becomes

$$F = C_1 r \cos(\phi_i \ln(r)) + C_2 r \sin(\phi_i \ln(r)) + C_3 r^2 + C_4 \qquad (B.58$$

B.2.4.3: $n>0$, $\beta_{ci}>0$

The roots in general are:

$$\begin{aligned} s_1 &= 1+\sqrt{\mu+\delta^{1/2}} = Re(s_1) \\ s_2 &= 1-\sqrt{\mu+\delta^{1/2}} = Re(s_2) \\ s_3 &= 1+\sqrt{\mu-\delta^{1/2}} \\ s_4 &= 1-\sqrt{\mu-\delta^{1/2}} \end{aligned} \qquad (B.59)$$

where:

$$\begin{aligned} \mu &= 1+n^2+\beta_{ci}\alpha_i^2/2 \\ \delta &= 4n^2+2n^2\beta_{ci}\alpha_i^2+\beta_{ci}^2\alpha_i^4/4 \end{aligned} \qquad (B.59.1)$$

The roots s_3 and s_4 are real if:

$$\mu^2 > \delta \qquad (B.60)$$

Expressing this in terms of n and β_{ci} yields:

$$n^2-1 > \beta_{ci} \qquad (B.61)$$

Thus, for $n^2-1 > \beta_{ci}$ the solution is:

$$F = r\left[C_1 r^{\sqrt{\mu+\delta^{1/2}}} + C_2 r^{-\sqrt{\mu+\delta^{1/2}}} + C_3 r^{\sqrt{\mu-\delta^{1/2}}} + C_4^{-\sqrt{\mu-\delta^{1/2}}}\right] \qquad (B.62)$$

and for $n^2-1 < \beta_{ci}$ we get:

$$F = r\left\{C_1 r^{\sqrt{\mu+\delta^{1/2}}} + C_2 r^{-\sqrt{\mu+\delta^{1/2}}} + C_3 \cos\left[(\mu-\delta^{1/2})^{1/2} \ln(r)\right] + C_4 \sin\left[(\mu-\delta^{1/2})^{1/2} \ln(r)\right]\right\} \qquad (B.63)$$

When $n^2-1 = \beta_{ci}$ we get:

$$F = r\left[C_1 r^{\sqrt{\mu+\delta^{1/2}}} + C_2 r^{-\sqrt{\mu+\delta^{1/2}}} + C_3 + C_4 \ln(r)\right] \qquad (B.64)$$

B.2.4.4: $n>0$, $\beta_{ci}<0$

In this case the roots are:

$$
\begin{aligned}
s_1 &= 1+\sqrt{\gamma+\delta^{1/2}}\\
s_2 &= 1-\sqrt{\gamma+\delta^{1/2}}\\
s_3 &= 1+\sqrt{\gamma-\delta^{1/2}}\\
s_4 &= 1-\sqrt{\gamma-\delta^{1/2}}
\end{aligned}
\qquad (B.65)
$$

and γ and $\delta^{1/2}$ are the same as in article B.2.4.3, but with a negative β_{ci}. The roots can be real or complex quantities. The roots s_1 and s_2 are real if:

$$\gamma+\delta^{1/2}>0 \qquad (B.66)$$

The roots s_3 and s_4 are real if:

$$\gamma-\delta^{1/2}>0 \qquad (B.67)$$

After some algebraic manipulation and substitution of n and β_{ci} we get:

$$1-n^2>|\beta_{ci}| \qquad (B.68)$$

which is also correct for $\gamma+\delta^{1/2}>0$
The solution for $1-n^2>|\beta_{ci}|$ is:

$$F=r\left[C_1 r^{\sqrt{\gamma+\delta^{1/2}}}+C_2 r^{-\sqrt{\gamma+\delta^{1/2}}}+C_3 r^{\sqrt{\gamma-\delta^{1/2}}}+C_4 r^{-\sqrt{\gamma-\delta^{1/2}}}\right] \qquad (B.6$$

and for $\gamma<-\delta^{1/2}$ the solution becomes:

$$F=r\left\{C_1\cos\left[(\gamma+\delta^{1/2})^{1/2}\ln(r)\right]+C_2\sin\left[(\gamma+\delta^{1/2})^{1/2}\ln(r)\right]+C_3 r^{\sqrt{\gamma-\delta^{1/2}}}\right.$$

$$\left.+C_4 r^{-\sqrt{\gamma-\delta^{1/2}}}\right\} \qquad (B.70)$$

If $\gamma<\delta^{1/2}$ the solution equals:

$$F=r\left\{C_1 r^{(\gamma+\delta^{1/2})^{1/2}}+C_2 r^{-(\gamma+\delta^{1/2})^{1/2}}+C_3\cos\left[(\gamma-\delta^{1/2})^{1/2}\ln(r)\right]+\right.$$

$$\left.C_4\sin\left[(\gamma-\delta^{1/2})^{1/2}\ln(r)\right]\right\} \qquad (B.71)$$

and for $\mu = -\delta^{1/2}$ the solution reads:

$$F = r\left[C_1 + C_2 \ln(r) + C_3 r^{\sqrt{\mu - \delta^{1/2}}} + C_4 r^{-\sqrt{\mu - \delta^{1/2}}}\right] \qquad (B.72)$$

For $\mu = \delta^{1/2}$ we get:

$$F = r\left[C_1 r^{\sqrt{\mu + \delta^{1/2}}} + C_2 r^{-\sqrt{\mu - \delta^{1/2}}} + C_3 + C_4 \ln(r)\right] \qquad (B.73)$$

Appendix C: The characteristic equation for a circular plate composed of annulars parts

The characteristic equation is obtained by employing the boundary conditions, and compatibility and equilibrium at the inner joints.

The conditions which must be satisfied at the inner joints are:

$$\text{at } r = r_1: \quad w_{1\,1} = w_{1\,2}$$

$$\frac{\partial w_{1\,1}}{\partial r} = \frac{\partial w_{1\,2}}{\partial r} \tag{C.1}$$

$$M_{rr_{1\,1}} = M_{rr_{1\,2}}$$

$$Q_{rr_{1\,1}} = \div Q_{rr_{1\,2}}$$

where

$$M_{rr} = \left[\frac{\partial^2 w}{mr^2} + \frac{\nu}{r} \left(\frac{\partial w}{\partial r} + \frac{1}{r} \frac{\partial^2 w}{\partial \theta^2} \right) \right] D$$

(C.1.1)

$$Q_{rr} = \left\{ \frac{\partial}{\partial r} \nabla^2 w + \frac{1 \mp \nu}{r} \frac{\partial}{\partial \theta} \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial w}{\partial \theta} \right) \right] \right\} D$$

The condition at the boundaries can be any one of the following: (See [9]).

(1) Clamped edge: $w = 0 \quad , \quad \frac{\partial w}{\partial r} = 0$ (C.2)

(2) Simply supported edge: $w = 0$

$$M_{rr} = 0 = \frac{\partial^2 w}{\partial r^2} = \frac{\nu}{r} \left(\frac{\partial w}{\partial r} + \frac{1}{r} \frac{\partial^2 w}{\partial \theta^2} \right) \tag{C.3}$$

(3) Free edge subjected to uniform compressive force p_o:

$$M_{rr} = 0 = \frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r} \left(\frac{\partial w}{\partial r} + \frac{1}{r} \frac{\partial^2 w}{\partial \theta^2} \right) \tag{C.4}$$

$$Q_{rr} = 0 = \frac{\partial}{\partial r} (\nabla^2 w) + \frac{1-\nu}{r} \frac{\partial}{\partial \theta} \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial w}{\partial \theta} \right) \right] + \frac{p_o}{D} \frac{\partial w}{\partial r} .$$

(4) Free to deflect but not to rotate:

$$\frac{\partial w}{\partial r} = 0 \tag{C.5}$$

$$Q_{rr} = 0 = \frac{\partial}{\partial r} (\nabla^2 w) + \frac{1-\nu}{r} \frac{\partial}{\partial \theta} \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial w}{\partial \theta} \right) \right] + \frac{p_o}{D} \frac{\partial w}{\partial r} .$$

A support, at an inner joints, is characterized by

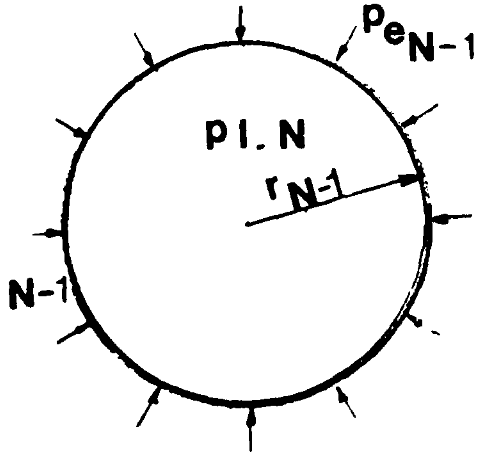
$$r = r_k: \qquad w_{k\ k-1} = w_{k\ k} = 0$$

$$\frac{\partial w_{k\ k-1}}{\partial r} = \frac{\partial w_{k\ k}}{\partial r} \tag{C.6}$$

$$M_{rr_{k\ k-1}} = M_{rr_{k\ k}} = \frac{\partial^2 w_{k\ k-1}}{\partial r^2} = \frac{\partial^2 w_{k\ k}}{\partial r^2}$$

In order to derive the characteristic equation, we start with the inner part, which can be either a hole or a continuous circular plate. The case of a plate with a hole is described by boundary conditions at the inner edge of the annular plate.

The case with an inner circular plate will be discussed next. The deflection equation for the circular part is described using $\beta_{ci} = 0$ (See Appendix B) for various n- values.



$$F_N(r) = C_{1N}J_n(u) + C_{2N}Y_n(u) + C_{3N}u^n\ C_{4N}u^{-n} \tag{C.7}$$

where $u = \alpha_N r$.

For $F(r)$ to be finite at $r = 0$, $C_{2N} = C_{3N} = 0$

hence

$$F_N(r) = C_{1N}J_n(u_N) + C_{4N}u^n \tag{C.8}$$

The solution of the outside annular plate is:

$$F_{N-1} = C_{1N-1}F_1(u_{N-1}) + C_{2\ N-1}F_2(u_{N-1}) + C_{3\ N-1}F_3(u_{N-1}) +$$

$$C_{4N-1}F_4(u_{N-1})\ , \quad \text{where} \quad u_{N-1} = \alpha_{N-1}r \tag{C.9}$$

If the boundary conditions at $r = r_{N-2}$ are:

$$w\ (r = r_{N-2}) = w_{N-1\ N-2}$$

$$\frac{\partial w}{\partial r}\bigg|_{r=r_{N-2}} = w_{,r_{N-1\ N-2}} \tag{C.10}$$

then using the B.C.'s at the mutual joint, N-1, we have:

$$\text{at } r = r_{N-1}: \quad w_{N\ N-1} = w_{N-1\ N-1} = 0 \tag{C.11}$$

$$\frac{\partial w_N}{\partial r}\Big|_{r=r_{N+1}} - \frac{\partial w_{N+1}}{\partial r}\Big|_{r=r_{N+1}} = 0 = \alpha_N \, w,_{r_{N}{}_{N+1}} + \alpha_{N-1} w,_{r_{N+1}{}_{N-1}} = 0$$

(C.12)

$$M_{rr_{N\;N-1}} + M_{rr_{N-1\;N+1}} = D_N \, (L_N^M w_N) \Big|_{r = r_{N+1}} + D_{N-1} \, (L_{N+1}^M w_{N-1}) \Big|_{r = r_{N-1}} = 0$$

$$Q_{N\;N+1} + Q_{N+1\;N-1} = D_N \, (L_N^Q w_N)\Big|_{r=r_{N+1}} + D_{N-1} (L_{N-1}^Q w_{N+1})\Big|_{r = r_{N-1}} = 0$$

where

$$L_N^M = \frac{\partial^2}{\partial r^2} + \frac{\nu}{r} \left(\frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial^2}{\partial \theta^2}\right) = \alpha_N^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\nu}{u} \frac{\partial}{\partial u} - \frac{\nu n^2}{u^2}\right.$$

$$L_N^Q = \frac{\partial}{\partial r} \nabla^2 w + \frac{1-\nu}{r} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial^2 w}{\partial \theta^2}\right) = \alpha_N^3 \left[\frac{\partial}{\partial u} \left(\frac{\partial^2}{\partial u^2} + \frac{1}{u} \frac{\partial}{\partial u} - \frac{n^2}{u^2}\right] w + \right. \qquad \text{(C.12.1)}$$

$$- \frac{1-\nu}{u} \frac{\partial}{\partial u} \left(\frac{1}{u} n^2\right) w$$

hence after some algebraic manipulation yields:

$$L_N^Q = \alpha_N^3 \left[\frac{\partial^3}{\partial u^3} + \frac{1}{u} \frac{\partial^2}{\partial u^2} - \frac{1}{u^2} \left((1-\nu) n^2 + 1 + n^2\right) \frac{\partial}{\partial n} + \frac{(3-\nu)n^2}{u^3} \right. \qquad \text{(C.12.2)}$$

Appendix D: <u>Manual for Buc.bas - buckling analysis of multi-annular plate (IBM - P.C Version)</u>

Insert in Drive A the Basic Compiler diskette.

Insert in Drive B the disk containing the Buc. exe. and the input file.

<u>Input file</u>

Line 1: Answer yes or no on the following questions:

1) Do you want intermediate printout?

2) Do you want normal mode printout?

3) Do you want a parametric study? (Two part only)

4) Is it buckling in tension (T) or compression (C)?

Ex: yes, yes, no, T.

Line 2: Input name of example

<u>Case a</u>: If answer on <u>question (3)</u> is <u>yes</u>, then:

Line 3: Input lowest and highest buckling modes. (Ex. 0,4)

Line 4: Input to questions: 1) Are there horizontal springs?

2) Are there vertical springs?

yes = 1 no = 2 Ex: 2, 2, (No springs)

Line 5: Input horizontal restraints, loading and springs (if answer for question in line 4 is yes) for each joint on separate line.

Line 5 - joint 0

Line 6 - joint 1.

Line 6 + npr + npr: Input (for each joint on separate line) horizontal restraints, loading and spring (if the answer is yes to first question in line 5)

Line 6 + npr x 3 Input (for joint on separate line) vertical b.c.'s and spring (if the answer is yes to second question in line 5).

<u>Example</u> (Parametric study- case a)

```
 in-plane b.c.        -----  ==========  -----
Forces:              ->                        <-
Vertical B.C         ^-----  ==========  -----^
                     0       1          1      0
                         1      2     2     1
```

1. no, no, yes, c (no printout, no printout of normal mode, parametric study (two part, compression force.
2. Example #1 (name of example)
3. 2,2 (No springs in horizontal and vertical directions)
4. 0,3 (lower mode 0, higher -3)
5. 0,1 (free to slide, $p_{e_0} = 1$ - joint 0
6. 2,0 (Continuous in horizontal direction, $p_{e_1} = 0$) - joint 1
7. 0 B.C. - simply-supported - joint 0
   - joint 1
8. 10 continuous
9. 1.5 0.3333, 1, 7 (outer radius - 1.5

   Poisson ratio = 1/3

   lowest E_2/E_1 related to 1 and highest modulus

   ratios - 7)

10. 1,9,1.1 (lowest r_1/r_o - 1, highest r_1/r_o - 9 and the factor multiplying p_{cr} (E_2/E_1, r_1/r_o-1) ← 1.1)

11. 10.1,0.5,1,2,10,100 (modulus ratios related to 1 to 7)

Explanation:

Line 7: Input vertical boundary conditions and spring if the answer to second question in Line 4 is yes.

Line 7 - joint 0

Line 8 - joint 1

Line 9: Input 1) Outer radius

2) Poison ratio (constant for all parts)

3) Lowest and highest integer (maximum 7) related to E_2/E_1.

Ex: 1.8 , 0.3333, 2, 7

Line 10: Input 1) Lowest and highest integers minimum/maximum 9 related to E_1/E_o.

2) A factor which multiplies the P_{cr} found for a lower r_1/r_o.

Line 11: Input modulus ratios, E_2/E_1 (No. of values must be equal to (highest-lowest) integer appearing in line 9.

Case b: If answer to question (3) on line 1 is no,

Line 3: Input 1) No. of part (No. of annular + circular parts) (maximum = 5)

2) No. of maximum terms in series (smaller or equal than 80.

Line 4: Input lowest and highest buckling modes.

Line 5: Input to questions: 1) Are there horizontal springs?

2) Are there vertical springs?

yes, =1, no = 2, Ex.: 2,2

Line 6: Input (for each joint on a separate line):

1) E- modulus of Elasticity

2) P_O- Poisson ratio

3) T - Thickness

} x npr

Line 6 + npr: Input (for each joint on a separate line): radius of joint

<u>Example (3 part plate)</u>

```
  in-plane b.c.      -----  =====  ----------  =====  -----

Forces:             ->                                     <-

Vertical B.C        ^-----  =====  ----------  =====  -----^
                    0      1      2          2      1      0
                       1      2      3    3      2      1
```

1) no, yes, no, c (no printout, print normal modes,

no parametric study, compression)

2) Example #2

3) 3, 50 (no of part=3, no of terms in each series = 50

4) 0,5 (lowest mode - 0, highest mode - 5)

5) 2,2 (no-spring in horizontal and vertical direction)

6) 1,0.333, 1 (E(1), P_O(1), T(1)) - plate 1

7) 1,0.1666,1 (E(2), P_O(2), T(2)) - plate 2

8) 1/2, 0.0, 1 (E(3), P_O(3),T(3)) - plate 3

9) 3 (r(0))

10) 1.5 (r(1))

11) 1.0 (r(2))

12) 0,1 (sliding, $p_{e_o} = 1.0$

13) 2,0 (continuous in horizontal dir., $p_{e_1} = 0$

14) 2,0 (continuous in horizontal dir., $p_{e_2} = 0$

15) 0 (simply supported)

16) 10 (continuous)

17) 10 (continuous)

Appendix E: Computer Program -Buc.bas

```
10 REM  Buckling analysis of circular plate composed of
   annular and circular part with different mechanical and
   geometrical properties.
20 REM Primary analysis is performed using stiffness
   equations starting at line 1000 and ends at 1380
30 INPUT "name of INPUT file ( name should include disk drive
   letter)";INPU$
40 OPEN INPU$ FOR INPUT AS #3
50  OPEN "lpt1:" AS #1
60 REM OPEN OUTP$ FOR OUTPUT AS #1
70 DEFDBL A -H:DEFDBL P-Z
80 SS1$="##.##":SS2$="###.###":SS3$="####.###":IPRI$="no"
90 DEFINT I-O
100 REM Questions: 1. Do you want intermediate printout--yes
                      or no,yepr$
                   2. Do you want normal mode printout--yes
                      or no,yenor$
                   3. Do you want parametric study (Tow
                      parts)-yes or no,yepar$
110 REM "Is it buckling in "tension"(T) or "compression"(C)"
120 INPUT #3,YEPR$,YENOR$,YEPAR$,YETY$
130 INPUT #3,NAMEC$:REM INPUT "name of example";NAMEC$
140 IF YEPAR$="yes" THEN NPR=2:NPL=NPR:NNS=80:GOTO 160
150 INPUT #3,NPR,NNS:NPL=NPR:REM input "no. of elements in
    the power series solution. No. of plates including
    holes";NNS,npr
160 INPUT #3,NOO,NCMAX:REM INPUT "The lowest and highest
    buckling mode (in circumferntial direction";nOO,NCMAX
170 INPUT #3,IYESH,IYESV:REM INPUT "are there any spring
    support in horizontal and vertical directions.
    Yesh=1,yesv=1,noh=2,nov=2";IYESH,IYESV
180 OPTION BASE 0
190 REM Dim declaration for the primary sub.
200 DIM E(5),PO(5),T(5),BETA(4),D(5):   REM starting from 1
210 NPLI=NPL-1:NBC=2+4*NPLI:NBC2=NBC*NBC:DIM R(4),IRES(4),
    PE(4),B(18),SA(4,4),S(4,4),SSL(4),SSR(4),U(4),AA (324)
    ,L(324),M(324):  REM -starting from 0 ,except b and a
220 IF IYESH=1 THEN DIM UK(4)
230 DIM SOF(5),SOCF(5),SRR(5,10),SOO(5,10),UU(5,10),RR(5,10):
    REM starting from 1
240 REM Dim declaration in lines 75-90 are for the bauckling
    analysis
250 DIM IBCW(4),IBCWG$(4),WR(4,4),DWR(4,4),BMR(4,4),SHR(4,4)
260 DIM WL(4,4),DWL(4,4),BML(4,4),SHL(4,4)
    ,BMAT(18,18),SBM(17,17),JOO (4)
    '  BMAt-starts from 1,1 to 4*npli+2,4*npli+2
270 DIM PCRR(7,9),SOFD(5),SOCFD(5),SRRR(4),SRRL(4),
    HJ(4),JH(4),IND(4)   ,DA(80),D2A(80),D3A(80),AAA(80),
    E22(7)
```

```
280 DIM GAMMA(5),ALAMDA(5),WW(5,10),DWW(5,10),BMW(5,10),
    SHW(5,10),SR( 5,4),SI(5,4),F(5,4),DF(5,4),D2F(5,4),
    D3F(5,4),D4F(5,4),A(5,4,80),SU(2),DSU(2),D2SU(2),D3SU(2)
    ,D4SU(2),G(2),DG(2),D2G(2),D3G(2), D4G(2),H(4),DH(4)
    ,D2H(4)  ,D3H(4),D4H(4)
290 IF IYESV=1 THEN DIM WK(4)
300 IF YEPAR$="yes" THEN GOTO 370
310 REM INPUT OF DATA
320 REM MECHNICAL AND GEOMETRICAL PROPPERTIES
330 FOR J=1 TO NPL:INPUT #3,E(J),PO(J),T(J):E(J)=E(J)*12*(1-
    PO(J)*PO(J))
340 D(J)=E(J)*T(J)^3/12/(1-PO(J)*PO(J)):NEXT J
350 FOR I=0 TO NPL-1:INPUT #3,R(I):NEXT I
360 FOR I=1 TO NPL-1:BETA(I)=R(I-1)/R(I):NEXT I
370 REM restraints and external forces
380 FOR I=0 TO NPL-1:INPUT #3, IRES(I),PE(I):IF IYESH=1 THEN
    INPUT #3, UK(I)
390 NEXT I
400 REM input of b.c. for vertical displacement
410 FOR I=0 TO NPLI:INPUT #3,IBCW(I):IF IYESV=1 THEN INPUT
    #3,WK(I)
420 IF IBCW(I)=0 THEN IBCWG$(I)=" ^" ELSE IF IBCW(I)=1 THEN
    IBCWG$(I)=" |" ELSE IF IBCW(I)=2 THEN IBCWG$(I)="||" ELSE
    IF IBCW(I)=3 THEN IBCWG$(I)="K^" ELSE IF IBCW(I)=4 THEN
    IBCWG$(I)="  "
430 IF IBCW(I)=10 THEN IBCWG$(I)="  " ELSE IF IBCW(I)=11 THEN
    IBCWG$(I)=" ^" ELSE IF IBCW(I)=12 THEN IBCWG$(I)=" |"
    ELSE IF IBCW(I)=13 THEN IBCWG$(I)="K^" ELSE IF IBCW(I)=14
    THEN IBCWG$(I)="||"
440 NEXT I:REM IBCW(I)=0 to 4 means edge b.c. IBCW(I)=10 to
    13 means b.c. at inn          -er joints
450 IF YEPAR$="no" THEN GOTO 530
460 REM "Data for a parametric study"
470 REM "TWO PLATE type  ----=====-----
480 REM "Parameters: 1)po(poisson ratio)-.3333,.3,.25,.1666,0
                     2)e(1)=1.0,e(2)/e(1)=.01,.1,.5,1.2
                       ,10,100
                     3)r(0)=1.0001,r(1)/r(0)=0.1 to
                       .9,dec=.1
490 T(1)=1!:T(2)=1!:E(1)=1
500 INPUT #3,R(0),POO,IE210,IE21F:PO(1)=POO:PO(2)=POO
510 INPUT #3,IR10S,IR10F,PFACTO
520 FOR I=IE210 TO IE21F:INPUT #3,E22(I):NEXT I
530 FOR NC=NOO TO NCMAX:IF YEPAR$="no" THEN GOTO 640
540 E(2)=1.1:IYEINTE=1:GOSUB 6350:E(2)=0
550 PRINT "----->>  nc=";NC:INPUT "p0---for nc";PCRO
560 FOR IE21=IE210 TO IE21F:E(2)=E22(IE21)
570 PRINT "pfact0=";PFACTO:INPUT "pfact0-new";PFACTO
580 CCD=12*(1-POO*POO):D(1)=E(1)*CCD*T(1)*T(1)*T(1)/12/(1-
    PO(1)*PO(1))   :D(2)=E(2)*CCD*T(2)*T(2)*T(2)/12/(1-
    PO(2)*PO(2))
590 PRINT #1,:PRINT #1,:PRINT #1,:PRINT
    #1,TAB(10);"r1/r0";TAB(25);"e2/e1=";:PRINT #1,USING
    SS3$;E(2):PRINT #1,
```

```
600 PRINT "e(2)=";E(2),"pcr0=";PCRO
610 PRINT "e(2)=";E(2),"pcr0=";PCRO
620 FOR IR10=IR10S TO
    IR10F:R(1)=IR10/10*R(0):BETA(1)=R(0)/R(1)
630 PRINT "r(1)=";R(1),"pcr0=";PCRO
640 REM Main program which calculates  the Pcr for each
    nc(no. of circuferential waves)
650 GOSUB 2250
660 IF YEPAR$="yes" THEN GOTO 690
670 PRINT "intial value of Pcr=P0*D0/r(npli))^2 ---input p0 -
    --for NC=";NC:INPUT P0:PCRO=P0*1!/R(0)/R(0):PRINT PCRO
680 GOSUB 1980:PCRR(0,0)=PCR1 :GOSUB 980:GOTO 1250
690 GOSUB 1980:PCRR(IE21,IR10)=PCR1:GOSUB 980:GOTO 1230
700 FOR I=1 TO NPLI:SOFD(I)=PCR*SOF(I)/D(I):IF
    ABS(SOF(I))<=.00001 AND ABS(SOCF(I))>.00001 THEN
    SOF(I)=0:GOTO 720
710 IF ABS(SOCF(I))<=.00001 AND ABS(SOCF(I)/SOF(I))<=.00001
    THEN SOCF(I)=0
720 SOCFD(I)=SOCF(I)*PCR/D(I):SRRR(I-1)=SRR(I,0)*PCR:
    SRRL(I)=PCR*SRR(I ,10):IF SOCFD(I)=0 THEN PCRSI=0 ELSE
    FCONST=NC*NC-1
725 IF NC=0 THEN FCONST2=0 ELSE IF NC=1 THEN FCONST2=-4 ELSE
    FCONST2=-4*NC*(NC-SQR(FCONST))
730 IF SOCFD(I)=0 THEN PCRSI=0:PCRSI2=0:GOTO 770
740 PCRSI=FCONST*D(I)/SOCF(I):PCRSI2=FCONST2*D(I)/SOCF(I):
    PRINT "pcrsi,pcrsi2=";PCRSI,PCRSI2:IF PCR=PCRSI THEN
    SOCFD(I)=FCONST ELSE IF PCR=PCRSI2 THEN SOCFD(I)=FCONST2
750 IF PCRSI<0 THEN PCRSIL=PCRSI*1.01:PCRSIR=.99*PCRSI
755 IF PCRSI2<0 THEN PCRSI2L=PCRSI2*1.01:PCRSI2R=.99*PCRSI2
760 IF PCRSI>0 THEN PCRSIL=PCRSI*.99:PCRSIR=1.01*PCRSI
765 IF PCRSI2>0 THEN PCRSI2L=PCRSI2*.99:PCRSI2R=1.01*PCRSI2
770 NEXT I
780 IF E(NPL)=0 THEN GOTO 790 ELSE SOFD(NPL)=- SRR(NPL,0)
    *PCR/D(NPL):SOCFD(NPL)=0:SRRR(NPLI)=PCR*SRR(NPL,0)
790 FOR I=1 TO NPLI:RI=R(I-1):IF SOCF(I)=0 THEN GOSUB
    1620:GOTO 810
800 GOSUB 3350
810 IF I-1=0 THEN INDE=0 ELSE INDE=0
820 FOR K=1 TO 4:GOSUB 6130
830 WR(I-1,K)=F(I,K):DWR(I-1,K)=DW:BMR(I-1,K)=BM:SHR(I-
    1,K)=SH+SRRR(I-1)*DW:NEXT K
840 RI=R(I):IF SOCF(I)=0 THEN GOSUB 1620:GOTO 860
850 GOSUB 3780
860 IF I=NPLI AND E(NPL)=0 THEN INDE=1 ELSE INDE=0
870 FOR K=1 TO 4:GOSUB 6130
880 WL(I,K)=F(I,K):DWL(I,K)=DW:BML(I,K)=BM:
    SHL(I,K)=SH+SRRL(I)*DW :NEXT K:NEXT I
890 IF E(NPL)=0 OR T(NPL)=0 THEN GOTO 950
900 I=NPL:RI=R(NPLI):GOSUB 1620:FOR KKW=1 TO 2 :IF KKW=1 THEN
    K=1 ELSE IF KKW=2 THEN K=4
910 IF KPROB<>0 THEN K=KKW
920 GOSUB 6130
930 WR(I-1,KKW)=F(I,K):DWR(I-1,KKW)=DW:BMR(I-1,KKW)=BM:SHR(I-
    1,KKW)=SH +SRRR(I-1)*DW:NEXT KKW
```

```
940 REM WR(I-1,2)=F(I,3):DWR(I-1,2)=DW:BMR(I-1,2)=BM:SHR(I-
    1,2)=SH+SRRR(I-1)*DW
950 GOSUB 1280
960 N=NBCC:GOSUB 5640
970 RETURN
980 REM "calculation of the Ci for the varios functions
990 IF YEPAR$="yes"  THEN GOTO 1000 ELSE IF YEPAR$="no" AND
    YENOR$="yes" THEN PRINT "nc=";NC;TAB(30);
    "Pcr=";PCRR(0,0)*R(0)*R(0)/1!;"*d0/r(0)^2":
    PCRR1=PCRR(0,0):GOTO 1030
1000 BEEP:PRINT "    nc=";NC:PRINT "     e(2)=";:PRINT USING
     SS2$;E(2):PRINT"    r(1)/r0=";:PRINT USING
     SS2$;IR10/10:PRINT "        Pcr=";:PRINT USING
     SS3$;PCRR(IE21,IR10)*R(0)*R(0)/1!;:PRINT "
     *d0/r(0)^2":PCRR1=PCRR(IE21,IR10)
1010 PRINT #1,TAB(9);:PRINT #1,USING SS2$;IR10/10;:PRINT
     #1,TAB(24);:PRINT #1,USING
     SS3$;PCRR(IE21,IR10)*R(0)*R(0)/1
1020 IF YENOR$="no" THEN RETURN
1030 FOR I=1 TO NBCC-1:FOR J=1 TO NBCC-1:SBM(I,J)=BMAT(I,J):
     NEXT J:B(I)=-BMAT(I,NBCC):NEXT I
1040 ISS=0:FOR I=1 TO NBCC-1:FOR J=1 TO NBCC-1:ISS=ISS+1:
     AA(ISS)=SBM(J,I):NEXT J:NEXT I:N=NBCC-1:GOSUB 2910:
     B(NBCC)=1!:GOTO 1050
1050 FOR I=1 TO NBCC:PRINT "b(";I;")=";B(I):NEXT I
1060 FOR I=1 TO NPLI:FOR JW=0 TO 10:RI=RR(I,JW):IF I=1 AND
     JW=0 THEN INDE=1 ELSE INDE=0
1070 IF I=NPLI AND JW=10 AND E(NPL)=0 THEN INDE=1 ELSE
     INDE=0
1080 IF SOCF(I)=0 THEN GOSUB 1620 ELSE GOSUB 3350
1090 INJ=4*(I-1):SUMW=0:SUMDW=0:SUMBM=0:SUMSH=0:FOR K=1 TO
     4:INJK=INJ+K:GOSUB
6130:SUMW=SUMW+B(INJK)*F(I,K):SUMDW=SUMDW+B(INJK)*DF(I,K):SUM
     BM=SU MBM+B(INJK)*BM:SUMSH=SUMSH+B(INJK)*SH:NEXT K
1100 WW(I,JW)=SUMW:DWW(I,JW)=SUMDW:BMW(I,JW)=SUMBM:
     SHW(I,JW)=SUMSH +SRR( I,JW)*PCRR1*SUMDW:NEXT JW:NEXT I
1110 IF E(NPL)=0 THEN GOTO 1160
1120 I=NPL:FOR JW=0 TO 5 :RI=RR(I,JW):IF JW=5 THEN RI=.00001
1130 IF SOCF(I)=0 THEN GOSUB 1620 ELSE GOSUB 3350
1140 INJ=4*NPLI:SUMW=0:SUMDW=0:SUMBM=0:SUMSH=0:FOR K=1 TO 4
     STEP 3:INJ=INJ+1:GOSUB
6130:SUMW=SUMW+B(INJ)*F(I,K):SUMDW=SUMDW+B(INJ)*DF(I,K):SUMBM
     =SUMB M+B(INJ)*BM:SUMSH=SUMSH+B(INJ)*SH:NEXT K
1150 WW(I,JW)=SUMW:DWW(I,JW)=SUMDW:BMW(I,JW)=SUMBM:
     SHW(I,JW)=SUMSH +SRR( I,JW)*PCRR1*SUMDW:NEXT JW
1160 WMAX=0:FOR I=1 TO NPLI:FOR J=1 TO 10:IF ABS(WW(I,J))>=
     ABS(WMAX) THEN WMAX=WW(I,J)
1170 NEXT J:NEXT I
1180 IF E(NPL)=0 THEN GOTO 1210
1190 I=NPL:FOR J=1 TO 5:IF ABS(WW(I,J))>=ABS(WMAX) THEN
     WMAX=WW(I,J)
1200 NEXT J
1210 GOSUB 7030
1220 RETURN
```

```
1230 PCRO=PFACTO*PCRR(IE21,IR10):NEXT IR10
1240 PCRO=PCRR(IE21,IR10S):NEXT IE21:GOSUB 6350
1250 NEXT NC
1260 CLOSE
1270 END
1280 REM Sub. which constructs the Buckling Matrix
1290 FOR I=0 TO NPLI:IF I=0 THEN GOTO 1450
1300 IROW=2+(I-1)*4:ICOLM=(I-1)*4:I1=IROW+1:I2=IROW+2:
     I3=IROW+3:I4=IROW +4:IF I=NPLI THEN KMAX=6 ELSE KMAX=8
1310 IF I=NPLI AND E(NPL)=0 THEN GOTO 1510
1320 IF IBCW(I)=10 THEN GOTO 1330 ELSE IF IBCW(I)=11 THEN
     GOTO 1360 ELSE IF IBCW(I)=12 THEN GOTO 1390 ELSE IF
     IBCW(I)=13 THEN GOTO 1420
1330 FOR K=1 TO KMAX:ICK=ICOLM+K:IF K>4 THEN GOTO 1340 ELSE
     BMAT(I1,ICK)=WL(I,K):BMAT(I2,ICK)=DWL(I,K):
     BMAT(I3,ICK)=BML(I ,K):B MAT(I4,ICK)=SHL(I,K):GOTO 1350
1340 K4=K-4:BMAT(I1,ICK)=-WR(I,K4):BMAT(I2,ICK)=-DWR(I,K4):
     BMAT(I3,ICK)  =-BMR(I,K4):BMAT(I4,ICK)=-SHR(I,K4)
1350 NEXT K:GOTO 1570
1360 FOR K=1 TO KMAX:ICK=ICOLM+K:IF K>4 THEN GOTO 1370 ELSE
     BMAT(I1,ICK)=WL(I,K):BMAT(I2,ICK)=0!:
     BMAT(I3,ICK)=DWL(I,K) :BM AT(I4 ,ICK)=BML(I,K):
     GOTO 1380
1370 K4=K-4:BMAT(I1,ICK)=0!:BMAT(I2,ICK)=WR(I,K4):
     BMAT(I3,ICK)=-DWR(I,K 4):BMAT(I4,ICK)=-BMR(I,K4)
1380 NEXT K:GOTO 1570
1390 FOR K=1 TO KMAX:ICK=ICOLM+K:IF K>4 THEN GOTO 1400 ELSE
     BMAT(I1,ICK)=WL(I,K):BMAT(I2,ICK)=0!:
     BMAT(I3,ICK)=DWL(I,K):BM AT(I4 ,ICK)=0!:GOTO 1410
1400 K4=K-4:BMAT(I1,ICK)=0!:BMAT(I2,ICK)=WR(I,K4):
     BMAT(I3,ICK)=0!:BMAT( I4,ICK)=DWR(I,K4)
1410 NEXT K:GOTO 1570
1420 FOR K=1 TO KMAX:ICK=ICOLM+K:IF K>4 THEN GOTO 1430 ELSE
     BMAT(I1,ICK)=WL(I,K):BMAT(I2,ICK)=DWL(I,K):
     BMAT(I3,ICK)=BML(I ,K):B MAT(I4,ICK)=SHL(I,K)-
     WK(I)*WL(I,K):GOTO 1440
1430 K4=K-4:BMAT(I1,ICK)=-WR(I,K4):BMAT(I2,ICK)=-DWR(I,K4):
     BMAT(I3,ICK)  =-BMR(I,K4):BMAT(I4,ICK)=-SHR(I,K4)
1440 NEXT K:GOTO 1570
1450 IF IBCW(0)=0 THEN GOTO 1460 ELSE IF IBCW(0)=1 THEN GOTO
1470 ELSE IF IBCW(0)=2 THEN GOTO 1480 ELSE IF IBCW(0)=3 THEN
     GOTO 1490 ELSE IF IBCW(0)=4 THEN GOTO 1500
1460 FOR K=1 TO 4:BMAT(1,K)=WR(0,K):BMAT(2,K)=BMR(0,K):NEXT
     K:GOTO 1570
1470 FOR K=1 TO 4:BMAT(1,K)=WR(0,K):BMAT(2,K)=DWR(0,K):NEXT
     K:GOTO 1570
1480 FOR K=1 TO 4:BMAT(1,K)=DWR(0,K):BMAT(2,K)=SHR(0,K):NEXT
     K:GOTO 1570
1490 FOR K=1 TO 4:BMAT(1,K)=BMR(0,K):BMAT(2,K)=SHR(0,K)-
     WK(I)*WR(0,K):NEXT K:GOTO 1570
1500 FOR K=1 TO 4:BMAT(1,K)=BMR(0,K):BMAT(2,K)=SHR(0,K):NEXT
     K:GOTO 1570
```

```
1510 IF IBCW(I)=11 THEN GOTO 1520 ELSE IF IBCW(I)=12 THEN
     GOTO 1530 ELSE IF IBCW(I)=13 THEN GOTO 1550 ELSE IF
     IBCW(I)=14 THEN GOTO 1540 ELSE IF IBCW(I)=10 THEN
     GOTO 1560
1520 FOR K=1 TO 4:BMAT(I1,ICOLM+K)=WL(I,K):
     BMAT(I2,ICOLM+K)=BML(I,K):NEXT K:GOTO 1570
1530 FOR K=1 TO 4:BMAT(I1,ICOLM+K)=WL(I,K):
     BMAT(I2,ICOLM+K)=DWL(I,K):NEXT K:GOTO 1570
1540 FOR K=1 TO 4:BMAT(I1,ICOLM+K)=DWL(I,K):
     BMAT(I2,ICOLM+K)=SHL(I,K):NEXT K:GOTO 1570
1550 FOR K=1 TO 4:BMAT(I1,ICOLM+K)=BML(I,K):
     BMAT(I2,ICOLM+K)=SHL(I,K)-WK(I)*WL(I,K ):NEXT K:
     GOTO 1570
1560 FOR K=1 TO 4:BMAT(I1,ICOLM+K)=BML(I,K):
     BMAT(I2,ICOLM+K)=SHL(I,K):NEXT K:GOTO 1570
1570 NEXT I
1580 IF E(NPL)=0 THEN NBCC=4*NPLI ELSE NBCC=2+4*NPLI
1590 ISS=0:FOR I=1 TO NBCC:FOR J=1 TO NBCC:ISS=ISS+1:
     AA(ISS)=BMAT(J,I):NEXT J:NEXT I
1600 IF YEPR$="no" THEN GOTO 1610 ELSE FOR I=1 TO NBCC:FOR
     J=1 TO NBCC:PRINT "bmat(";I;",";J")=";BMAT(I,J)
     ,;:NEXT J:PRINT:NEXT I
1610 RETURN
1620 REM SUB. "solves the D.E. for a uniform state of in-
     plane stresses"- srr=-sof(i)*pcrr(i)
1630 KPROB=0:SR(I,1)=2+NC:SR(I,2)=2-NC:SR(I,3)=-NC:
     SR(I,4)=NC :SI(I,1)=0 :SI(I,2)=0:SI(I,3)=0:SI(I,4)=0
1640 PRINT;TAB(10);"  i   ";TAB(30);"  sr   ";TAB(45);"  si
     ";PRINT
1650 FOR K=1 TO 4:PRINT;TAB(12);I;TAB(30);SR(I,K)
     ;TAB(45);SI(I,K):NEXT K:PRINT
1660 K=1:JUP=NNS*2:NNS2=2*NNS
1670 A(I,K,0)=1:FOR J=2 TO JUP STEP 2:J2=J/2
1680 DENOM2=J+SR(I,K):DENOM=DENOM2*DENOM2-NC*NC
1690 A(I,K,J2)=-SOFD(I)*1/DENOM*A(I,K,J2-1)
1700 REM   PRINT "i=";I,"k=";K,"j=";J,"a(i,k,j)=";A(I,K,J2)
1710 NEXT J:IF K=2 THEN RETURN
1720 KPP=K:J0=0:GOSUB
4730:F(I,K)=SUM:DF(I,K)=DSUM:D2F(I,K)=D2SUM:D3F(I,K)=D3SUM:D4
     F(I,K )=D4SUM:GOSUB 6290: IF YEPR$="yes" THEN PRINT
     "F(I,K),K  ";F(I,K),K
1730 K=2:IF NC>0 THEN GOTO 1770 ELSE EPSS=.0000001:
     SR(I,2)=SR(I,1)+.0000001:JUP=NNS2:A(I,K,0)=1!:
     GOSUB 1670
1740 KPP=K:J0=0:JMAX=NNS:GOSUB 4730
1750 FE1=SUM:DFE1=DSUM:D2FE1=D2SUM:D3FE1=D3SUM:D4FE1=D4SUM
1760 F(I,2)=(FE1-F(I,1))/EPSS:DF(I,2)=(DFE1-DF(I,1))/EPSS:
     D2F(I,2)=(D2F E1-D2F(I,1))/EPSS:D3F(I,2)=(D3FE1-
     D3F(I,1))/EPSS:D4F(I,2)=(D4 FE1-D4F(I,1))/EPSS
     :GOSUB 6290:IF YEPR$="no" THEN GOTO 1910 ELSE
     PRINT"f(i,k),k";F(I,K),K:GOTO 1910
1770 K=2:IF NC=1 THEN C2=1!:C1=-C2*SOFD(I)/A(I,1,0)/4/(1+NC)
     :A(I,K,1)=1!:A(I,K,0)=0:GOTO 1820
1780 K=2:IF NC=2 THEN A(I,K,0)=1!:A(I,K,1)=0:C2=-SOFD(I)/4:
     A(I,K,2)=1!:C1=-C2*SOFD(I)/4/ (1+NC):GOTO 1820
```

```
1790 K=2:JUP=2*NC-4:GOSUB 1670
1800 J=2*NC-2:J2=J/2:C2=A(I,K,J2-1)*SOFD(I)*((NC-2)*(NC-2)-
     NC*NC)/NC/(N C-1)/8:A(I,K,J2)=0
1810 J=2*NC:J2=J/2:C1=-C2*SOFD(I)/4/(1+NC)/A(I,1,0):
     A(I,K,J2)=1!
1820 FOR J=2*NC+2 TO NNS2 STEP 2:J1=J-NC*2:JJ2=J1/2:J2=J/2
1830 DAA1=A(I,1,JJ2)*(J1+SR(I,1)-1)*(4*(J1+SR(I,1)
     )*(J1+SR(I,1)-2)-4*NC *NC-SOCFD(I)*2)
1840 DAA2=2*SOFD(I)*(J1+SR(I,1)-2)*A(I,1,JJ2-1):
     DA1=DAA1+DAA2
1850 ANOM=(J-NC)*(J-NC)-NC*NC:DENOM=(J+2-NC)*(J+2-NC)-
     NC*NC:A(I,K,J2)=-(C1*DA1/ANOM+A(I,K,J2-1)*SOFD(I))
     /DENOM:NEXT J
1860 SMUL=1:FOR J=1 TO NC:SMUL=SMUL*RI:NEXT J
1870 F(I,3)=C1*F(I,1)+C2*SMUL:
     DF(I,3)=C1*DF(I,1)+C2*NC*SMUL/RI:
     D2F(I,3)=C1*D2F(I,1)+SMUL*NC*(NC-1)*C2/RI/RI
1880 D3F(I,3)=C1*D3F(I,1)+C2*NC*(NC-1)*(NC-2)*SMUL/RI/RI/RI:
     D4F(I,3)=D4 F(I,1)*C1+C2*NC*(NC-1)*(NC-2)*(NC-3)*
     SMUL/RI/RI/RI/RI
1890 K=2:JMAX=NNS:KPP=K:GOSUB 4730
1900 K=2:KKK=3:IDS=1:GOSUB 5050
1910 IF RI=0 THEN GOTO 1930 ELSE IF NC=0 THEN GOTO 1920 ELSE
     K=3:POW=-NC:GOSUB 1950:GOTO 1940
1920 IDS=1:SUM=0:D2SUM=0:DSUM=0:D3SUM=0:D4SUM=0:
     K=3:KKK=4:F(I,KKK)=1:DF (I,KKK)=0:D2F(I,KKK)=0:
     D3F(I,KKK)=0:D4F(I,KKK)=0:GOSUB 5050:GOTO 1940
1930 K=3:F(I,K)=0:DF(I,K)=0:D2F(I,K)=0:D3F(I,K)=0:
     D4F(I,K)=0:GOSUB 6290:IF YEPR$="no" THEN GOTO 1940 ELSE
     PRINT"f(i,k),k";F(I,K),K
1940 K=4:POW=NC:GOSUB 1950:RETURN
1950 F(I,K)=RI^POW:DF(I,K)=POW*RI^(POW-1):D2F(I,K)=POW*(POW-
     1)*RI^(POW-2):D3F(I,K)=POW*(POW-1)*(POW-2)*RI^(POW-3)
1960 D4F(I,K)=POW*(POW-1)*(POW-2)*(POW-3)*RI^(POW-4)
1970 GOSUB 6290:IF YEPR$="no" THEN RETURN ELSE PRINT"f(i,k),k
     ";F(I,K),K:RETURN
1980 REM Sub. which calculates the Pcr using Newton-Raphson
     iterative algoritmus
1990 EPS=.00001:IEND=30:IER=0:DPCR=.00001:ISING=0
2000 PCR=PCR0:GOSUB 700:DET0=DET:BEEP:PRINT"det,pcr",DET,PCR
2010 IF ABS(DET)<=.000001 THEN PCR1=PCR:RETURN
2020 FOR IIIT=1 TO IEND
2030 IF PCR0>=PCRSIL AND PCR0<=PCRSIR THEN ISING=1+ISING:GOTO
2180
2035 IF PCR0>=PCRSI2L AND PCR0<=PCRSI2R THEN
     ISING=1+ISING:GOTO 2235
2040 PCR=PCR0+DPCR:GOSUB 700:DDET=(DET-DET0)/DPCR:IF DDET=0
     THEN GOTO 2170 ELSE PCR1=PCR0-DET0/DDET
2050 IF  PCR1<0  THEN PCR1=PCR0/2
2060 PCR=PCR1:GOSUB
700:DET1=DET:BEEP:PRINT"pcr1,det1",PCR1,DET1
2070 IF (ABS((PCR1))>1 AND ABS((PCR1-PCR0)/PCR1)<=100*EPS)
     AND ABS(DET1)<=EPS THEN RETURN
```

```
2080 IF (ABS(PCR1)<=1 AND ABS(PCR1-PCRO)<=100*EPS) AND
     ABS(DET1)<=EPS THEN RETURN
2090 IF (ABS(PCR1)<=ABS(PCRSI) AND ABS(PCRO)>=ABS(PCRSI)) OR
     (ABS(PCR1)>=ABS(PCRSI) AND ABS(PCRO)<=ABS(PCRSI)) THEN
     ISING=ISING+1:GOTO 2180
2100 IF DET1*DETO<0 THEN PCR=(DET1*PCRO-DETO*PCR1)/(DET1-
     DETO):GOSUB 700:PCR1=PCR:DET1=DET:IF ABS(DET)<=.000001
     THEN RETURN
2110 IF (ABS(PCR1)>=.95*ABS(PCR20) AND
     ABS(PCR1)<=1.05*ABS(PCR20)) AND ABS(DET1)>=.01  THEN
     PCR1=(PCR1+PCRO)/2:PCR=PCR1:GOSUB 700:DET1=DET:BEEP:
     PRINT "Bi-section meth.-pcr1,det1 ";PCR1,DET1
2120 PCR20=PCRO:PCRO=PCR1:DETO=DET1
2130 BEEP:BEEP:PRINT"pcr0=";PCRO,"det0=";DETO,"iiit=";IIIT:IF
     ABS(PCRO)<=.05 THEN GOTO 2240
2140 IF (ABS(PCR1-PCRO)<=.01 AND ABS(DET1)<=.01) AND
     IIIT=IEND THEN RETURN
2150 NEXT IIIT
2160 PRINT "there is no convergance after ";IEND;"
     loops":GOTO 2240
2170 PRINT "the derivative is equal to <<<< zero >>> . Try
     another intial Pcr":GOTO 2240
2180 PCRO=PCRSI:PCR=PCRO:GOSUB 700:DETO=DET:BEEP:
     PRINT"detsi,pcrsi",DET,PCR
2190 IF ABS(DET)<=.000001 THEN PCR1=PCR:RETURN
2200 IF ISING=1 THEN  GOTO 2230
2210 IF ABS(PCRO)>=1 THEN PCRO=3*PCRSI ELSE IF PCRSI<0 THEN
     PCRO=-.55+PCRO ELSE PCRO=.55+PCRO
2220 GOTO 2000
2230 BEEP:BEEP:BEEP:PRINT "Pcr is in PCRSI=";PCRSI;"
     neighborhood----<<<<< please enter your guess
     ---at least far from pcrsi---":INPUT PCRO:GOTO 1980
2235 BEEP:BEEP:BEEP:PRINT "Pcr is in PCRSI2=";PCRSI2;"
     neighborhood----<<<<< please enter your guess
     ---at least far from pcrsi---":INPUT PCRO:GOTO 1980
2240 INPUT "<<<<< input your intial guess for the case
     >>>>";PCRO:GOTO 1980
2250 REM primary analysis using stiffness equations
2260 REM stiffness matrix
2270 FOR I=0 TO NPL-1
2280 IF I=NPL-1 GOTO 2320
2290 SSR(I)=E(I+1)*BETA(I+1)*(1+PO(I+1)+(1-PO(I+1))
     /BETA(I+1)^2)/R(I+1)/(BETA(I+1)^2-1)/(1-PO(I+1)^2)
2300 S(I+1,I)=E(I+1)*BETA(I+1)*2/R(I+1)/(BETA(I+1)^2-1)/(1-
     PO(I+1)^2):  SA(I+1,I)=S(I+1,I)
2310 IF I=0 THEN S(0,0)=-SSR(I):GOTO 2370
2320 SSL(I)=E(I)*(1+PO(I)+BETA(I)^2*(1-
     PO(I)))/R(I)/(BETA(I)^2-1)/(1-PO (I)^2)
2330 S(I-1,I)=E(I)*2/R(I)/(BETA(I)^2-1)/(1-PO(I)^2):SA(I-
     1,I)=S(I-1,I)
2340 IF I=NPL-1 GOTO 2350 ELSE GOTO 2360
2350 SSR(NPL-1)=E(NPL)/R(NPL-1)/(1-PO(NPL))
2360 S(I,I)=-SSR(I)-SSL(I)
2370 SA(I,I)=S(I,I):NEXT I
```

```
2380 FOR I=0 TO NPL-1:B(I+1)=PE(I):NEXT I
2390 FOR J=0 TO NPL-1
2400 IF IRES(J)=0 OR IRES(J)=2 THEN GOTO 2430
2410 FOR I=0 TO NPL-1:SA(I,J)=0:SA(J,I)=0:NEXT I
2420 SA(J,J)=S(J,J):B(J+1)=0
2430 NEXT J
2440 ISS=0:FOR I=0 TO NPL-1:FOR J=0 TO NPL-1:ISS=ISS+1:
     AA(ISS)=SA(J,I):NEXT J:NEXT I
2450 REM solution of the deformation
2460 N=NPL:GOSUB 2910
2470 FOR I=0 TO NPL-1:U(I)=B(I+1):NEXT I
2480 IF YEPR$="yes" THEN GOSUB 3160
2490 REM calculation of stresses and deformation
2500 FOR I=1 TO NPL-1
2510 SOF(I)=E(I)*(1+PO(I))*(U(I)-BETA(I)*U(I-1))/R(I)
     /(BETA(I)^2-1)/(1-PO(I)^2)
2520 SOCF(I)=BETA(I)*E(I)*(1-PO(I))*(-BETA(I)*U(I)+U(I-
     1))*R(I)/(BETA(I )^2-1)/(1-PO(I)^2)
2530 DEC=(R(I-1)-R(I))/10
2540 FOR J=0 TO 10
2550 RR(I,J)=R(I-1)-DEC*J
2560 SRR(I,J)=-SOF(I)+SOCF(I)/RR(I,J)/RR(I,J)
2570 SOO(I,J)=-SOF(I)-SOCF(I)/RR(I,J)/RR(I,J)
2580 UU(I,J)=(U(I-1)*BETA(I)*(1-R(I)^2/RR(I,J)^2)+U(I)*(R(I-
     1)^2/RR(I,J )^2-1))*RR(I,J)/R(I)/(BETA(I)^2-1)
2590 IF I>0 AND J=0 AND ABS(SRR(I,0)-SRR(I-1,10))<=.0000001#
     THEN SRR(I,0)=SRR(I-1,10)
2600 NEXT J:NEXT I
2610 REM stresses and deformation in circular part
2620 IF E(NPL)=0 THEN GOTO 2690
2630 FOR J=0 TO 5
2640 SRR(NPL,J)=E(NPL)*U(NPL-1)/R(NPL-1)/(1-PO(NPL)):IF J=0
     AND ABS(SRR(NPL,J)-SRR(NPLI,10))<=.0000001# THEN
     SRR(NPL,0)=SRR(NPLI,10)
2650 SOO(NPL,J)=SRR(NPL,J)
2660 RR(NPL,J)=R(N-1)-R(N-1)*J/5
2670 UU(NPL,J)=U(N-1)/R(I-1)*RR(NPL,J)
2680 NEXT J
2690 REM printout of stresses
2700 IF YEPR$="no" THEN RETURN
2710 IF E(NPL)=0 THEN NPLMAX=NPL-1 ELSE NPLMAX=NPL
2720 FOR I=1 TO NPLMAX
2730 PRINT:PRINT "Plate no. :        ";I:PRINT:PRINT:PRINT :IF
     I=NPL GOTO 2760
2740 SSP$="###.###":PRINT TAB(20);"no(";I;")=";:PRINT USING
     SSP$;SOF(I):PRINT TAB(20);"noc(";I;")=";:PRINT USING
     SSP$;SOCF(I)
2750 IF SOF(I)=0 THEN GOTO 2760 ELSE PRINT TAB(20);
     "betac(";I;")=" ;:PRINT USING SSP$;SOCF(I)/SOF(I)
2760 IF I=NPL THEN SJ=5 ELSE SJ=10
2770 PRINT:PRINT:PRINT TAB(10);"  rr   ";TAB(25);"  nrr
     ";TAB(45);"  noo   ";TAB(63);"  uu   ":PRINT:PRINT
2780 FOR J=0 TO SJ
```

```
2790 SSP$="###.###":PRINT TAB(9);:PRINT USING SSP$;RR(I,J);:
     PRINT TAB(25);:PRINT USING SSP$;SRR(I,J);:PRINT
     TAB(45);:PRINT USING SSP$;SOO(I,J);:PRINT TAB(62);:PRINT
     USING SSP$;UU(I,J)
2800 NEXT J:NEXT I
2810 FOR I=1 TO NPLMAX
2820 PRINT #1,:PRINT #1,"Plate no. :        ";I:PRINT
     #1,:PRINT #1,:PRINT #1,:IF I=NPL GOTO 2850
2830 SSP$="###.###":PRINT #1, TAB(20);"no(";I;")=";:PRINT #1,
     USING SSP$;SOF(I):PRINT #1, TAB(20);"noc(";I;")="
     ;:PRINT #1, USING SSP$;SOCF(I)
2840 IF SOF(I)=0 THEN GOTO 2850 ELSE PRINT #1,TAB(20);
     "betac(";I;")=";:PRINT #1, USING SSP$;SOCF(I)/SOF(I)
2850 IF I=NPL THEN SJ=5 ELSE SJ=10
2860 PRINT #1,:PRINT #1,:PRINT #1, TAB(10);"  rr  ";TAB(25);"
     nrr  ";TAB(45);"  noo  ";TAB(63);"  uu  ":PRINT #1,
     :PRINT #1,
2870 FOR J=0 TO SJ
2880 SSP$="###.###":PRINT  #1, TAB(9);:PRINT #1, USING
     SSP$;RR(I,J);:PRINT #1, TAB(25);:PRINT #1, USING
     SSP$;SRR(I,J);:PRINT #1, TAB(45);:PRINT #1, USING
     SSP$;SOO(I,J);:PRINT #1, TAB(62);:PRINT #1, USING
     SSP$;UU(I,J)
2890 NEXT J:NEXT I
2900 RETURN
2910 REM sub. for solving simulteneous equations(SIMQ)
2920 TOL=0:KS=0:JJ=-N
2930 FOR J=1 TO N:JY=J+1:JJ=JJ+N+1:BIGA=0:IT=JJ-J
2940 FOR I=J TO N:IJ=IT+I:BA=ABS(BIGA)-ABS(AA(IJ))
2950 IF BA<0 THEN GOTO 2960 ELSE GOTO 2970
2960 BIGA=AA(IJ):IMAX=I
2970 NEXT I
2980 BA=ABS(BIGA)-TOL:IF BA<=0 THEN GOTO 2990 ELSE GOTO 3000
2990 KS=1:RETURN
3000 I1=J+N*(J-2):IT=IMAX-J
3010 FOR K=J TO N:I1=I1+N:I2=I1+IT:SAFE=AA(I1):
     AA(I1)=AA(I2):AA(I2)=SAFE
3020 AA(I1)=AA(I1)/BIGA:NEXT K
3030 SAFE=B(IMAX):B(IMAX)=B(J):B(J)=SAFE/BIGA:JN=J-N
3040 IF JN<>0 THEN GOTO 3050 ELSE GOTO 3110
3050 IQS=N*(J-1)
3060 FOR IX=JY TO N:IXJ=IQS+IX:IT=J-IX
3070 FOR JX=JY TO N:IXJX=N*(JX-1)+IX:JJX=IXJX+IT:
     AA(IXJX)=AA(IXJX)-AA(IXJ)*AA(JJX)
3080 NEXT JX
3090 B(IX)=B(IX)-B(J)*AA(IXJ):NEXT IX
3100 NEXT J
3110 NY=N-1:IT=N*N
3120 FOR J=1 TO NY:IA=IT-J:IB=N-J:IC=N
3130 FOR K=1 TO J:B(IB)=B(IB)-AA(IA)*B(IC):IA=IA-N:IC=IC-
     1:NEXT K
3140 NEXT J
3150 RETURN:    'END OF SUBROUTINE
3160 REM printout of result
```

```
3170 IPRI$="yes":GOSUB 6390
3180 PRINT:PRINT:PRINT" <<<<<<<<<  data and results
     >>>>>>>>>>>>":PRINT
3190 PRINT #1,:PRINT #1,:PRINT #1," <<<<<<<<<  data and
     results  >>>>>>>>>>>>":PRINT #1,
3200 PRINT TAB(5);"plate no. ";TAB(18);"  e  ";TAB(30);"  po
     ";TAB(45);"  th  ";TAB(62);" beta ":PRINT
3210 PRINT #1, TAB(5);"plate no. ";TAB(18);"  e  ";TAB(30);"
     po  ";TAB(45);"  th  ";TAB(62);" beta ":PRINT  #1,
3220  FOR I=1 TO NPL:PRINT TAB(9);I;TAB(16);:PRINT USING
     "###.###";E(I);:PRINT  TAB(31);:PRINT USING
     "##.##";PO(I);:PRINT TAB(42);:PRINT USING
     "###.###";T(I);
3230  IF I=NPL THEN GOTO 3240 ELSE PRINT TAB(60);:PRINT USING
     "###.###";BETA(I)
3240  PRINT #1, TAB(9);I;TAB(16);:PRINT #1, USING
     "###.###";E(I);:PRINT  #1, TAB(31);:PRINT #1, USING
     "##.##";PO(I);:PRINT #1, TAB(42);:PRINT #1, USING
     "###.###";T(I);
3250  IF I=NPL THEN GOTO 3260 ELSE PRINT #1, TAB(60);:PRINT
     #1, USING  "###.###";BETA(I)
3260 NEXT I
3270 PRINT:PRINT TAB(5);"joint no.";TAB(18);" r ";TAB(30);"
     res ";TAB(45);" pe ";TAB(62);" u ":PRINT
3280 PRINT #1,:PRINT #1, TAB(5);"joint no.";TAB(18);" r
     ";TAB(30);" res ";TAB(45);" pe ";TAB(62);" u ":PRINT
     #1,
3290  FOR I=0 TO NPL-1:PRINT TAB(9);I;TAB(16);:PRINT USING
     "###.###";R(I);:PRINT TAB(31);IRES(I);TAB(42);:PRINT
     USING "###.###";PE(I);:PRINT TAB(60);:PRINT USING
     "###.###";U(I)
3300  PRINT #1, TAB(9);I;TAB(16);:PRINT #1, USING
     "###.###";R(I);:PRINT #1, TAB(31);IRES(I);TAB(42);:
     PRINT #1, U SING "###.###";PE(I);:PRINT #1,TAB(60);
     :PRINT #1, USING  "###.###";U(I)
3310 NEXT I
3320 REM INPUT " 1 to continue";YCON
3330 PRINT:PRINT:PRINT:PRINT
3340 PRINT #1,:PRINT #1,:PRINT #1,:PRINT #1,:RETURN
3350 REM Soultion of Annular buckling Equation .The D.E.
     being solved is:
3360 REM D.E.=ri^4*(d4f)+2*ri^3*(d3f)-ri^2*(1+2*nc^2+
     betac(i)*(alfa(i)^2)-r i^2*alfa(i)^2)*(d2f)+
     ri*(1+2*nc^2+betac(i)*alfa(i)^2-ri^2*alfa(i)^2)*(
     d1f)-(nc^2*(4-nc^2+betac(i)*alfa(i)^2
     +ri^2*alfa(i)^2)*(d0f)=0
3370 ' VAR: RI=r(i-1) or r(i)
3380 '       di=d^i/dr^i
             f=f(r), NC=no. of waves in circumferntial
             direction
             alfa(i)=sqr(-sof(i)/d(i)) if sof(i)<0 (tension)
```

```
3390 '       alfa(i)=sqr(sof(i)/d(i)) if sof(i)>0
             (compression)
             d(i)=e(i)*t(i)^3/12/(1-po(i)^2)
             betac(i)=socf(i)/sof(i)  betac(i)>0 means
             compression
3400 '       betac(i)<0 means tension
             betac(i)*alfa(i)^2=socf(i)/d(i)=sofcd(i)
3410 '   D.E.=ri^4*(d4f)+2*ri^3*(d3f)-ri^2*(1+2*nc^2+socfd(i)-
              ri^2*sofd(i)^ 2)*(d2f)+ri*(1+2*nc^2+socfd(i)-
              ri^2*sofd(i)*(d1f)-(nc^2*(4-nc^2+socfd(i)+
              ri^2*sofd(i))*(d0f)=0
3420 '        IF SOFD(I)>=0 THEN ALFA(I)=SQR(SOFD(I))
3430 ' Assuming a power series solution:
      f(i,k,ri)=sum[j=0,nn]{a(i,k,j)*ri(j+sr(i,k)+i*si(i,k))
      ;[k=1 to 4]
3440 ' calculation of the roots. The indicial eq. is:
      I.E.=(s-1)^4-(2+2*nc^2+socfd)*(s-1)^2+((nc-1)^2+
           socfd*(1-nc^2))=0
3450 ' The roots are calculated in the follwing subroutine
3460 GAMMA(I)=1+NC*NC+SOCFD(I)/2:
     ALAMDA(I)=4*NC*NC+2*NC*NC*SOCFD(I )+SOC FD(I)*SOCFD(I)
     /4:REM alamda(i)=(2*NC^2+SOCFD(I)/2)^2-4*NC^2*(NC^2-1)
3470 IF ALAMDA(I)>0 THEN GOTO 3480 ELSE IF ALAMDA(I)=0 THEN
     GOTO 3590 ELSE GOTO 3640
3480 ' <<<< lamda>0  >>>>>>
3490 PHI1=GAMMA(I)+SQR(ALAMDA(I)):PHI2=GAMMA(I)-
     SQR(ALAMDA(I))
3500 IF ABS(PHI1-CINT(PHI1))<=1E-10 THEN PHI1=CINT(PHI1)
3510 IF ABS(PHI2-CINT(PHI2))<=1E-10 THEN PHI2=CINT(PHI2)
3520  IF PHI1>0 THEN PRE1=SQR(PHI1):PIM1=0
3530  IF PHI1=0 THEN PRE1=0:PIM1=0
3540  IF PHI1<0 THEN PRE1=0!:PIM1=SQR(-PHI1)
3550  IF PHI2>0 THEN PRE2=SQR(PHI2):PIM2=0
3560  IF PHI2=0 THEN PRE2=0:PIM2=0
3570  IF PHI2<0 THEN PRE2=0!:PIM2=SQR(-PHI2)
3580 GOTO  3670
3590 ' <<<<< lamda=0  >>>>>>>
3600 PHI1=GAMMA(I):PHI2=GAMMA(I)
3610 IF PHI1>0 THEN PRE1=SQR(PHI1):PIM1=0:
     PRE2=PRE1:PIM2=PIM1
3615 IF PHI1=0 THEN PRE1=0:PIM1=0:PRE2=PRE1:PIM2=PIM1
3620 IF PHI1<0 THEN PRE1=0!:PIM1=SQR(-
     PHI1):PRE2=PRE1:PIM2=PIM1
3630 GOTO 3670
3640 '   <<<<<<< lamda<0  >>>>>>>>>
3650 PRE1=SQR((GAMMA(I)+SQR(GAMMA(I)*GAMMA(I)+
     ABS(ALAMDA(I))))/2):  PIM1= SQR(-ALAMDA(I))/2/PRE1
3660 PRE2=-PRE1:PIM2=PIM1
3670 REM Derivation of the roots
3680 SR(I,1)=1+PRE1:SI(I,1)=PIM1
3690 SR(I,2)=1-PRE1:SI(I,2)=-PIM1
3700 SR(I,3)=1+PRE2:SI(I,3)=PIM2
3710 SR(I,4)=1-PRE2:SI(I,4)=-PIM2
```

```
3720 IF (PRE2=-PRE1 AND PIM2=PIM1) AND PIM2<>0 THEN
     SR(I,1)=1+PRE1:SI(I,1)=PIM1:SR(I,2)=1+PRE1:SI(I,2)=-
     PIM1:SR(I,3)=1 +PRE2:SI(I,3)=PIM2:SR(I,4)=1+PRE2:
     SI(I,4)=-PIM2
3730 REM FOR J=1 TO 4:BIGA=SR(I,J):FOR K=J TO 4
3740 REM IF BIGA<=SR(I,K) THEN GOTO 2343 ELSE
     BIGA=SR(I,K):SR(I,K)=SR(I,J):SI(I,K)=SI(I,J)
3750 REM  K:SR(I,J)=BIGA:NEXT J
3760 PRINT;TAB(10);"  i   ";TAB(30);"  sr  ";TAB(45);"  si
     ":PRINT
3770 FOR K=1 TO 4:PRINT;TAB(12);I;TAB(30);SR(I,K);
     TAB(45);SI(I,K):NEXT K:PRINT
3780 ' Calculation of the various solutions
3790 K=1:IF SR(I,1)<= 0 OR SR(I,1)>0  AND SI(I,1)=0 THEN GOTO
3800 ELSE GOTO 3810
3800 GOSUB 6000:J00(1)=J0:J02=J0/2:A(I,1,J02)=1:GOSUB
     4650:GOTO 3820
3810 A(I,1,0)=1!:A(I,2,0)=0!:GOSUB 4260:GOTO 3880
3820 K=2:  '  f(i,2)--will be consider for s=sr(i,2) only
3830 IF SR(I,2)<>SR(I,1) THEN GOTO 3840 ELSE GOTO 3860
3840 GOSUB 6000:J00(2)=J0:IF J0>0 AND INDX>0 THEN GOTO 3870
     ELSE GOTO 3850
3850 J0=J00(INDX):J02=J0/2:A(I,2,J02)=1:GOSUB 4650:IF
     ABS(F(I,2)-F(I,1))<1E-10 THEN GOTO 3860 ELSE GOTO 3880
3860 J0=J00(1)/2:A(I,2,J0)=1:IDS=1:KKK=1:GOSUB 5220:GOTO
     3880
3870 IF INDX>1 THEN GOTO 3850 ELSE
     IDS=1:KKK=INDX:J0=J00(KKK):GOSUB 4840:GOTO 3880
3880 K=3:IF ABS(SI(I,3))>0 THEN GOTO 4010
3890 IF SR(I,3)=SR(I,1) AND SR(I,3)=SR(I,2) THEN GOTO 3990
     ELSE IF SR(I,3)=SR(I,1) THEN KKK=1 ELSE IF
     SR(I,3)=SR(I,2)  THEN KKK=2 ELSE GOTO 3910
3900 GOTO 4000
3910 GOSUB 6000:J00(3)=J0:IF J0>0 AND INDX>0 THEN GOTO 3950
     ELSE  GOTO 3920
3920 J0=J00(KKK)/2:A(I,3,0)=1!:GOSUB 4650:IF ABS(F(I,3)-
     F(I,1))<=1E-10 THEN KKK=1:GOTO 3980
3930 IF ABS(F(I,3)-F(I,2))<=1E-10 THEN KKK=2: GOTO 3980
3940 GOTO 4040
3950 IF IND(1)>0 THEN KKK=1:GOTO 3980
3960 IF IND(2)>0 THEN KKK=3:GOTO 3980
3970 IF IND(3)>0 OR IND(4)>0 THEN KKK=3:GOTO 3920
3980 J0=J00(KKK)/2:A(I,3,J0)=1:IDS=1:GOSUB 4840:GOTO 4040
3990 KKK=1:J0=J00(1)/2:A(I,3,0)=1:IDS=2:GOSUB 5220:GOTO 4040
4000 J0=J00(KKK)/2:A(I,3,J0)=1:IDS=1:GOSUB 5220:GOTO 4040
4010 IF SR(I,3)=SR(I,1) AND SI(I,3)=SI(I,1) THEN GOTO 4030
4020 A(I,3,0)=1:A(I,4,0)=0:GOSUB 4260:GOTO 4220
4030 A(I,3,0)=1:A(I,4,0)=0:GOSUB 4260:GOTO 4220
4040 K=4: 'f(i,4 will be considered only for s=sr(i,4) only
4050 IF (SR(I,4)<>SR(I,1)) AND (SR(I,4)<>SR(I,2)) AND
     (SR(I,4)<>SR(I,3)) THEN GOTO 4110
4060 IF SR(I,4)=SR(I,1) AND SR(I,4)=SR(I,2) AND
     SR(I,4)=SR(I,3)  THEN GOTO 4180
```

```
4070 IF SR(I,4)=SR(I,1) THEN KKK=1 ELSE IF SR(I,4)=SR(I,2)
     THEN KKK=2 ELSE IF SR(I,4)=SR(I,3) THEN KKK=3 ELSE GOTO
     4090
4080 GOTO 4200
4090 IF (SR(I,4)=SR(I,1) AND SR(I,4)=SR(I,2))  THEN KKK=1
     ELSE IF (SR(I,4)=SR(I,3) AND SR(I,4)=SR(I,1)) THEN KKK=1
     ELSE IF (SR(I,4)=SR(I,2) AND SR(I,4)=SR(I,3)) THEN KKK=2
     ELSE GOTO 4220
4100 GOTO 4190
4110 GOSUB 6000:JOO(4)=JO:IF JO>0 AND INDX>0 THEN GOTO 4120
     ELSE 4160
4120 FOR III=1 TO 3:IF IND(III)>0 THEN KKK=III:GOTO 4150
4130 NEXT III
4140 IF IND(4)>0 THEN KKK=4:GOTO 4160
4150 A(I,4,0)=1!:IDS=1:GOSUB 4840:GOTO 4220
4160 JO=JOO(4)/2:A(I,4,JO)=1!:GOSUB 4650:FOR III=1 TO 3:IF
     ABS(F(I,4)-F(I,III))<=1E-10 THEN KKK=III:GOTO 4210
4170 NEXT III:GOTO 4220
4180 JO=JOO(1)/2:A(I,4,JO)=1!:IDS=3:KKK=1:GOSUB 5220:GOTO
     4220
4190 JO=JOO(KKK)/2:A(I,4,JO)=1!:IDS=2:GOSUB 5220:GOTO 4220
4200 JO=JOO(KKK)/2:A(I,4,JO)=1!:IDS=1:GOSUB 5220:GOTO 4220
4210 JO=JOO(KKK)/2:A(I,4,JO)=1!:IDS=1:GOSUB 5220:GOTO 4220
4220 IF YEPR$="no" THEN RETURN ELSE PRINT;TAB(5);"  i
     ";TAB(20);"   fi   ";TAB(35);" dfi   ";TAB(50);"  d2 i
     ";TAB(65);"  d3fi   ":PRINT
4230 FOR K=1 TO 4:PRINT;TAB(5);I;TAB(17);F(I,K);TAB(30);
     DF(I,K);TAB(47);D2F(I ,K);T AB(65);D3F(I,K):NEXT
     K:PRINT
4240 REM FOR K=1 TO 4:PRINT #2,"i,k,F(I,K),D2F(I,K),
     D3F(I,K),d4f(i,k)
     ";I,K,F(I,K),DF(I,K),D2F(I,K),D3F(I,K),D4F(I,K):
     NEXT K:PRINT
4250 RETURN
4260 REM Solution with s=sr+i*si(Reccurrence formula)
4270 NNS2=2*NNS:FOR J=2 TO NNS2 STEP 2:J2=J/2
4280 ANOMR1=J-2+SR(I,K):ANOMI1=SI(I,K)
4290 ANOMR=ANOMR1*ANOMR1-ANOMI1*ANOMI1-
     NC*NC:ANOMI=2*ANOMR1*ANOMI1
4300 IF (ANOMR=0 AND ANOMI=0) OR SOFD(I)=0 THEN
     A(I,K,J2)=0:A(I,K+1,J2)=0:GOTO 4380
4310 DENOMR2=J-1+SR(I,K):DENOMI2=SI(I,K)
4320 TIR1=DENOMR2*DENOMR2-
     DENOMI2*DENOMI2:TII1=2*DENOMR2*DENOMI2
4330 TIR2=TIR1*TIR1-TII1*TII1:TII2=2*TII1*TIR1
4340 DENOMR=TIR2-2*GAMMA(I)*TIR1+(1-NC*NC)*(1-
     NC*NC+SOCFD(I)):DENOMI=TI I2-2*GAMMA(I)*TII1:
     DNOR=DENOMR*DENOMR+DENOMI*DENOMI
4350 ARR=-SOFD(I)*(DENOMR*ANOMR+DENOMI*ANOMI)/DNOR
4360 AII=-SOFD(I)*(DENOMR*ANOMI-DENOMI*ANOMR)/DNOR
4370 A(I,K,J2)=ARR*A(I,K,J2-1)-AII*A(I,K+1,J2-
     1):A(I,K+1,J2)=AII*A(I,K, J2-1)+ARR*A(I,K+1,J2-1)
4380 NEXT J
4390 K=K:JMAX=NNS:KPP=K:JO=0:GOSUB 5370
```

```
4400 SU(1)=SUM:DSU(1)=DSUM:D2SU(1)=D2SUM:D3SU(1)=D3SUM:
     D4SU(1)=D4S UM
4410 K=K+1:JMAX=NNS:KPP=K:JO=0:GOSUB 5370
4420 SU(2)=SUM:DSU(2)=DSUM:D2SU(2)=D2SUM:D3SU(2)=D3SUM:
     D4SU(2)=D4S UM
4430 K=K-1:G(1)=COS(SI(I,K)*LOG(RI)):
     G(2)=SIN(SI(I,K)*LOG(RI))
4440 DG(1)=-SI(I,K)*G(2)/RI:DG(2)=SI(I,K)*G(1)/RI
4450 D2G(1)=-SI(I,K)*DG(2)/RI+SI(I,K)*G(2)/RI/RI
4460 D2G(2)=SI(I,K)*DG(1)/RI-SI(I,K)*G(1)/RI/RI
4470 D3G(1)=-SI(I,K)*D2G(2)/RI+2*SI(I,K)*DG(2)/RI/RI-
     2*SI(I,K)*G(2)/RI/ RI/RI
4480 D3G(2)=SI(I,K)*D2G(1)/RI-
     2*SI(I,K)*DG(1)/RI/RI+2*SI(I,K)*G(1)/RI/R I/RI
4490 D4G(1)=-SI(I,K)*D3G(2)/RI+3*SI(I,K)*D2G(2)/RI/RI-
     6*SI(I,K)*DG(2)/R I/RI/RI+6*SI(I,K)*G(2)/RI/RI/RI/RI
4500 D4G(2)=SI(I,K)*D3G(1)/RI-
     3*SI(I,K)*D2G(1)/RI/RI+6*SI(I,K)*DG(1)/RI /RI/RI-
     6*SI(I,K)*G(1)/RI/RI/RI/RI
4510 HS=0:FOR IH=1 TO 2:FOR JJH=1 TO 2:HS=HS+1
4520 H(HS)=SU(IH)*G(JJH)
4530 DH(HS)=DSU(IH)*G(JJH)+SU(IH)*DG(JJH)
4540 D2H(HS)=D2SU(IH)*G(JJH)+2*DSU(IH)*DG(JJH)+
     SU(IH)*D2G(JJH)
4550 D3H(HS)=D3SU(IH)*G(JJH)+3*D2SU(IH)*DG(JJH)+
     3*DSU(IH)*D2G(JJH)   +SU(I H)*D3G(JJH)
4560 D4H(HS)=D4SU(IH)*G(JJH)+4*D3SU(IH)*DG(JJH)+
     6*D2SU(IH)*D2G(JJH )+4*DSU(IH)*D3G(JJH)+
     SU(IH)*D4G(JJH)
4570 NEXT JJH:NEXT IH
4580 F(I,K+1)=H(1)-H(4):F(I,K)=H(2)+H(3)
4590 DF(I,K+1)=DH(1)-DH(4):DF(I,K)=DH(2)+DH(3)
4600 D2F(I,K+1)=D2H(1)-D2H(4):D2F(I,K)=D2H(2)+D2H(3)
4610 D3F(I,K+1)=D3H(1)-D3H(4):D3F(I,K)=D3H(2)+D3H(3)
4620 D4F(I,K+1)=D4H(1)-D4H(4):D4F(I,K)=D4H(2)+D4H(3)
4630 GOSUB 6290:K=K+1:GOSUB 6290:K=K-1:IF YEPR$="no" THEN
     RETURN ELSE PRINT"k,k+1,f(i,k),f(i,k+1)
     ";K,K+1,F(I,K),F(I,K+1)
4640 RETURN
4650 REM Reccurrence formula and solution  for s=sr
4660 NNS2=2*NNS:FOR J=2+JO TO NNS2 STEP 2:J2=J/2
4670 ANOM1=J+SR(I,K)-2:ANOM=ANOM1*ANOM1-NC*NC:IF ANOM=0 OR
     SOFD(I)=0 THEN A(I,K,J2)=0:GOTO 4710
4680 SS=J+SR(I,K):SS1=J+SR(I,K)-1:SS2=J+SR(I,K)-
     2:SS3=J+SR(I,K)-3:SNN=1+2*NC*NC+SOCFD(I):
     DENOM=SS*SS1*SS2*SS3+2*SS*SS1*SS2-SNN*SS*SS 1+SNN*SS-
     NC*NC*(4-NC*NC+SOCFD(I))
4690 IF A(I,K,J2-1)<>0 THEN A(I,K,J2)=-
     SOFD(I)*ANOM/DENOM*A(I,K,J2-1) ELSE A(I,K,J2)=0
4700 REM DENOM1=J+SR(I,K)-1:DENOM=(DENOM1*DENOM1-
     PHI1)*(DENOM1*DENOM1-PHI2)   :IF A(I,K,J2-1)<>0 THEN
     A(I,K,J2)=-SOFD(I)*aNOM/DENOM*A(I,K,J2-1)   ELSE
     A(I,K,J2)=0
```

```
4710  NEXT J:REM PRINT  ANOM,DENOM:PRINT
      "i=";I,"k=";K,"j=";J,"a(i,k,j2)=";A(I,K,J2):NEXT J
4720 GOTO 4830:REM PRINT
     #2,"a(";I;",";K;",";JO")=";A(I,K,JO):GOTO 4080
4730   JMAX=NNS:GOSUB 5370:RETURN
4740 NNS2=2*NNS:SUM=0:DSUM=0:D2SUM=0:D3SUM=0:D4SUM=0:FOR J=JO
     TO NNS2 STEP 2:J2=J/2
4750 SUM=SUM+A(I,K,J2)*RI^(J+SR(I,KPP))
4760 DSUM=DSUM+A(I,K,J2)*(J+SR(I,KPP))*RI^(J+SR(I,KPP)-1)
4770 D2SUM=D2SUM+A(I,K,J2)*(J+SR(I,KPP))*(J+SR(I,KPP)-
     1)*RI^(J+SR(I,KPP )-2)
4780 D3SUM=D3SUM+A(I,K,J2)*(J+SR(I,KPP))*(J+SR(I,KPP)-
     1)*(J+SR(I,KPP)-2 )*RI^(J+SR(I,KPP)-3)
4790 D4SUM=D4SUM+A(I,K,J2)*(J+SR(I,KPP))*(J+SR(I,KPP)-
     1)*(J+SR(I,KPP)-2 )*(J+SR(I,KPP)-3)*RI^(J+SR(I,KPP)-4)
4800 REM   PRINT "k,kpp,SUM,DSUM,D2SUM,D3SUM,J";K,KPP,SUM,
     DSUM,D2SUM,D3SUM,D4S UM,J
4810 NEXT J
4820 RETURN
4830 KPP=K:GOSUB
4730:F(I,K)=SUM:DF(I,K)=DSUM:D2F(I,K)=D2SUM:D3F(I,K)=D3SUM:D4
     F(I,K )=D4SUM:GOSUB 6290: IF YEPR$="no" THEN RETURN ELSE
     PRINT  "F(I,K),K  ";F(I,K),K:RETURN
4840 REM Reccurrence formula and solution using
     D^ndsF/DS^nds.  Nds=no. of differentiation
4850 NNS2=2*NNS:FOR J=2+JO TO NNS2 STEP 2:J2=J/2
4860 ANOM1=J+SR(I,KKK)-2:ANOM=ANOM1*ANOM1-NC*NC:IF ANOM=0 OR
     SOFD(I)=0 THEN
     AAA(J2)=0:DA(J2)=0:D2A(J2)=0:D3A(J2)=0:GOTO 4930
4870 DNOM=2*NOM1:D2NOM=2:D3NOM=0
4880 DENOM2=J+SR(I,KKK)-1:DENOM1=DENOM2^2-
     GAMMA(I):DENOM=1/(DENOM1^2-AL AMDA(I))
4890 DDENOM=2*DENOM1*2*DENOM2:D2DENOM=4*(2*DENOM2^2+DENOM1)
     :D3DENO M=4*( 4*DENOM2+2*DENOM2):AAA(J2)=NOM*DENOM
4900 DA(J2)=(DNOM*DENOM-NOM*DDENOM*DENOM^2):IF IDS=1 THEN
     GOTO 4930
4910 D2A(J2)=(D2NOM*DENOM-2*DNOM*DDENOM*DENOM^2-
     NOM*D2DENOM*DENOM^2+2*N OM*DDENOM^2*DENOM^3):IF IDS=2
     THEN GOTO 4930
4920 D3A(J2)=(D3NOM*DENOM-3*D2NOM*DDENOM*DENOM^2-
     3*DNOM*D2DENOM*DENOM^2
     +6*DNOM*DDENOM^2*DENOM^3+6*NOM*D2DENOM*DDENOM*DENOM^3-
     NOM*D3D ENOM*DENOM^2-6*NOM*DDENOM^3*DENOM^4)
4930 NEXT J:REM PRINT  "j,nom,nom1,denom,da(j2),aaa(j2)
     ";J,NOM,NOM1,DENOM,DA(J2),AAA(J2):NEXT J
4940 JO2=JO+2:JO22=JO2/2:A(I,K,JO22)=-SOFD(I)*DA(JO22):IF
     IDS=2 THEN A(I,K,JO22)=-SOFD(I)*D2A(JO22) ELSE IF IDS=3
     THEN A(I,K,JO22)=-D3A(JO22)*SOFD(I)
4950 NNS2=2*NNS:FOR J=4+JO TO NNS2 STEP
     2:J2=J/2:SUMD=0:SUM2D=0:SUM3D=0:FOR LL=JO+2 TO J STEP
     2:LL2=LL/2:IF AAA(LL2)=0 AND DA(LL2)=0 AND D2A(LL2)=0
     AND D3A(LL2)=0 THEN GOTO 5030
4960 SUMD=SUMD+DA(LL2)/AAA(LL2):IF IDS=1 THEN GOTO 4990
4970 SUM2D=SUM2D+D2A(LL2)/DA(LL2):IF IDS=2 THEN GOTO 4990
```

```
4980 SUM3D=SUM3D+D3A(LL2)/D2A(LL2)
4990 NEXT LL
5000 A(I,K,J2)=SUMD*A(I,KKK,J2):IF IDS=1 THEN GOTO 5030:'
     a=da/ds
5010 A(I,K,J2)=SUM2D*A(I,KKK,J2):IF IDS=2 THEN GOTO 5030:'
     a=d2a/d2s
5020 A(I,K,J2)=SUM3D*A(I,KKK,J2)                          :' A
     =D2A/D2S
5030 NEXT J:REM   PRINT
     "da(j2),aaa(j2),j";DA(J2),AAA(J2),J:PRINT
     "i=";I,"k=";K,"j=";J,"a(i,k,j2)=";A(I,K,J2):NEXT J
5040 J02=J0/2:A(I,K,J0)=0:KPP=KKK:GOSUB 4730
5050 IF RI<=.000001  THEN B0=0 ELSE B0=LOG(RI)^IDS
5060 IF RI=<.000001 OR IDS-1<0  OR LOG(RI)=0 THEN B1=0 ELSE
     B1=LOG(RI)^(IDS-1)
5070 IF  RI=<.000001 OR IDS-2<0  OR LOG(RI)=0 THEN B2=0 ELSE
     B2=LOG(RI)^(IDS-2)
5080 IF RI<=.000001 OR IDS-3<0  OR LOG(RI)=0 THEN B3=0 ELSE
     B3=LOG(RI)^(IDS-3)
5090 IF RI<=.000001 OR IDS-4<0  OR LOG(RI)=0 THEN B4=0 ELSE
     B4=LOG(RI)^(IDS-4)
5100 F(I,K)=F(I,KKK)*B0+SUM
5110 DF(I,K)=DF(I,KKK)*B0+DSUM+F(I,KKK)*IDS*B1/RI
5120 D2G=IDS*(IDS-1)*B2/RI/RI+IDS*B1*(-1/RI)/RI
5130 D3G=IDS*(IDS-1)*(IDS-2)*B3/RI/RI/RI-3*IDS*(IDS-
     1)*B2/RI/RI/RI+IDS* B1*2/RI^3
5140 D4G=IDS*(IDS-1)*(IDS-2)*(IDS-3)*B4/RI^4-6*IDS*(IDS-
     1)*(IDS-2)*B3/R I^4+11*IDS*(IDS-1)*B2/RI^4-
     IDS*B1*6/RI^4
5150 D2F(I,K)=D2F(I,KKK)*LOG(RI)^IDS+2*DF(I,KKK)
     *IDS*B1/RI+F(I,KKK )*D2G +D2SUM
5160 D3F(I,K)=D3F(I,KKK)*LOG(RI)^IDS+3*D2F(I,KKK)
     *IDS*B1/RI+3*DF(I ,KKK)  *D2G+F(I,KKK)*D3G+D3SUM
5170 D4F(I,K)=D4F(I,KKK)*LOG(RI)^IDS+
     4*D3F(I,KKK)*IDS*B1/RI+6*D2F( I,KKK
     )*D2G+4*DF(I,KKK)*D3G+F(I,KKK)*D4G+D4SUM
5180 GOSUB 6290
5190 IF YEPR$="no" THEN RETURN
5200  PRINT "K,F(I,K),DF(I,K),D2F(I,K),D3F(I,K),
     d4f(i,k),KKK,B0,B1,B2,B3, B4,DG ,D2G,D3G,d4g";K
     ,F(I,K),DF(I,K),D2F(I,K),D3F(I,K),D4F(I,K),KKK
     ,B0,B1,B2,B3,B4,DG,D2G,D3G,D4G:PRINT "-------RI=";RI
5210 RETURN
5220 REM Solution for equal roots
5230 NNS2=2*NNS:FOR J=0 TO NNS2 STEP 2:J2=J/2
5240 DAA1=A(I,KKK,J2)*(J+SR(I,KKK)-
     1)*(4*(J+SR(I,KKK))*(J+SR(I,KKK)-2)-4*NC*NC-SOCFD(I)*2)
5250 IF J=0 THEN GOTO 5260 ELSE DAA2=2*SOFD(I)*(J+SR(I,KKK)-
     2)*A(I,KKK,J2-1):DA1=DAA1+DAA2
5260 ANOM1=J+SR(I,K)-2:ANOM=ANOM1*ANOM1-NC*NC
5270 SS=J+SR(I,K):SS1=J+SR(I,K)-1:SS2=J+SR(I,K)-
     2:SS3=J+SR(I,K)-3:SNN=1+2*NC*NC+SOCFD(I):
     DENOM=SS*SS1*SS2*SS3+2*SS*SS1*SS2-SNN*SS*SS 1+SNN*SS-
     NC*NC*(4-NC*NC+SOCFD(I))
```

```
5280 IF J>2 THEN GOTO 5330
5290 IF J=0 AND DAA1=0 THEN GOTO 5340 ELSE IF J=0 AND DAA1<>0
     THEN PRINT "TRY ADDITIONAL TERMS":BEEP:BEEP:BEEP:STOP
5300 IF J=2 AND (DENOM=0 AND ANOM=0) THEN PRINT "try
     additional terms":BEEP:BEEP:BEEP:STOP
5310 IF J=2 AND DENOM=0 THEN A(I,K,0)=-
     DA1/SOFD(I)/ANOM:A(I,K,J2)=1!
5320 IF J=2 AND DENOM<>0 THEN A(I,K,0)=0:A(I,K,J2)=-
     DA1/DENOM
5330 A(I,K,J2)=-(DA1+A(I,K,J2-1)*SOFD(I)*ANOM)/DENOM
5340 NEXT J
5350 KPP=K:GOSUB 4730
5360 GOSUB 5050:RETURN
5370 REM Sub.  Evaluate a polynom at r=ri using nested
     procedure
5380 REM VAR:jmax,j0,a(i,k,j2),k,kpp
5390 SUM=0:DSUM=0:D2SUM=0:D3SUM=0:D4SUM=0:JJ=JMAX
5400 JJ2=JJ*2:A0=A(I,K,JJ):SJS=JJ2+SR(I,KPP):
     DA0=SJS*A0:D2A0=DA0*( SJS-1 ):D3A0=D2A0*(SJS-
     2):D4A0=D3A0*(SJS-3)
5410 SUM=SUM*RI*RI+A0:DSUM=DSUM*RI*RI+DA0:
     D2SUM=D2SUM*RI*RI+D2A0:D3SUM=D3SUM*RI*RI+D3A0:
     D4SUM=D4SUM*RI*RI+D4A0
5420 REM PRINT "k,kpp,SUM,DSUM,D2SUM,D3SUM,d4sum,Jj";
     K,KPP,SUM,DSUM,D2SUM,D3 SUM,D 4SUM,JJ
5430 IF JJ=<0 THEN GOTO 5450
5440 JJ=JJ-1:GOTO 5400
5450 IF SR(I,KPP)<0 THEN SPO=1/RI^ABS(SR(I,KPP)) ELSE
     SPO=RI^SR(I,KPP)
5460 SUM=SPO*SUM:DSUM=SPO*DSUM/RI:D2SUM=SPO*D2SUM/RI/RI
     :D3SUM=SPO* D3SUM /RI/RI/RI:D4SUM=SPO*D4SUM/RI/RI/RI/RI
5470 RETURN
5480 REM data for a run
5490 'DATA 1,.2,1
5500 ' DATA 1,.2,1
5510 ' DATA 1,.2,1
5520 ' DATA 1,.2,1
5530 ' DATA 4,3,2,1
5540 ' DATA 2,0,0,1,0,0,0,0
5550 ' data for two part plate'
5560 DATA "aaa"
5570 DATA 2,30,6
5580 DATA 2,2
5590 DATA 1,.3000000,1.
5600 DATA 1,.3000000,1.
5610 DATA 1.00001,.500
5620 DATA 2,1,0,0
5630 DATA 0,10
5640 REM Invert of a matrix
5650 'Var.: aA(n)=Input matrix destroyed in computation and
     replaced by resultant invrese (columnwise)
5660 REM     N=   order of matrix A.(not columnwise)
5670 '       det= resultant determinant
5680 '       L=   WORK VECTOR OF LENGTH N
```

```
5690 '        M=   WORK VECTOR OF LENGTH N
5700 '        dim a(n),rl(n),rm(n)
5710 DET=1!:NK=-N
5720 FOR K=1 TO N:NK=NK+N:L(K)=K:M(K)=K:KK=NK+K:BIGA=AA(KK)
5730 FOR J=K TO N:IZ=N*(J-1):FOR I=K TO N:IJ=IZ+I
5740 IF (ABS(BIGA)-ABS(AA(IJ)))>=0 THEN GOTO 5750 ELSE
     BIGA=AA(IJ):L(K)=I:M(K)=J
5750 NEXT I:NEXT J
5760 J=L(K):IF (J-K)<=0 THEN GOTO 5780 ELSE KI=K-N
5770 FOR I=1 TO N:KI=KI+N:HOLD=-AA(KI):JI=KI-
     K+J:AA(KI)=AA(JI):AA(JI)=HOLD:NEXT I
5780 I=M(K):IF (I-K)<=0 THEN GOTO 5800
5790 JP=N*(I-1):FOR J=1 TO N:JK=NK+J:JI=JP+J:HOLD=-
     AA(JK):AA(JK)=AA(JI):AA(JI)=HOLD:NEXT J
5800 IF BIGA<>0 THEN GOTO 5810 ELSE DET=0!:RETURN
5810 FOR I=1 TO N:IF (I-K)=0 THEN GOTO 5820 ELSE
     IK=NK+I:AA(IK)=AA(IK)/(-BIGA)
5820 NEXT I
5830 FOR I=1 TO N:IK=NK+I:HOLD=AA(IK):IJ=I-N
5840 FOR J=1 TO N:IJ=IJ+N:IF (I-K)=0 THEN GOTO 5850 ELSE IF
     (J-K)=0 THEN GOTO 5850 ELSE KJ=IJ-
     I+K:AA(IJ)=HOLD*AA(KJ)+AA(IJ)
5850 NEXT J:NEXT I
5860 KJ=K-N:FOR J=1 TO N:KJ=KJ+N:IF (J-K)<>0 THEN
     AA(KJ)=AA(KJ)/BIGA
5870 NEXT J
5880 DET=DET*BIGA
5890 AA(KK)=1/BIGA
5900 NEXT K
5910 K=N
5920 K=K-1
5930 IF K=<0 THEN GOTO 5990 ELSE I=L(K)
5940 IF (I-K)<=0 THEN GOTO 5960 ELSE JQ=N*(K-1):JR=N*(I-1)
5950 FOR J=1 TO N:JK=JQ+J:HOLD=AA(JK):JI=JR+J:AA(JK)=-
     AA(JI):AA(JI)=HOLD:NEXT J
5960 J=M(K):IF (J-K)<=0 THEN GOTO 5920 ELSE KI=K-N
5970 FOR I=1 TO N:KI=KI+N:HOLD=AA(KI):JI=KI-K+J:AA(KI)=-
     AA(JI):AA(JI)=HOLD:NEXT I
5980 GOTO 5920
5990 RETURN
6000 REM Sub. for checking case of denom=0(s=sr only)
6010 ' VAR: pre1,pre2,hj1,hj2,hj3,hj4,jh1,jh2,jh3,jh4,
     ind1,ind2,ind3,ind4
6020 HJ(1)=1-SR(I,K)-PRE2:JH(1)=CINT(HJ(1)):HJ(2)=1-
     SR(I,K)+PRE2:JH(2)= CINT(HJ(2))
6030 HJ(3)=1-SR(I,K)-PRE1:JH(3)=CINT(HJ(3)):HJ(4)=1-
     SR(I,K)+PRE1:JH(4)= CINT(HJ(4))
6040 FOR III=1 TO 4:NOM2=(2-SR(I,K)-2)^2-NC^2
6050 IF ABS(HJ(III)-JH(III))<=1E-08 AND(JH(III)>0 AND
     JH(III)/2=INT(JH(III)/2)) AND NOM2<>0  THEN
     JO=JH(III):L(III)=JH(III):IND(III)=III:GOTO 6070
6060 IND(III)=0:JO=0:L(III)=0
6070 NEXT III
```

```
6080 IF IND(1)<>0 OR IND(2)<>0 OR IND(3)<>0 OR IND(4)<>0 THEN
     GOTO 6090 ELSE GOTO 6110
6090 INDX=1:BIGA=L(1):FOR III=1 TO 4:IF BIGA<L(III) THEN
     BIGA=L(III):INDX=II
6100 NEXT III:JO=BIGA
6110 IF YEPR$="no" THEN RETURN
6120 PRINT JO,INDX,IND(1),IND(2),IND(3),IND(4):RETURN
6130 REM calculation of internal forces at r=ri
6140 ' var: f(i,k),df(i,k),d2f(i,k),d3f(i,k),ri
6150 DW=DF(I,K)
6160 BM=-(D2F(I,K)+PO(I)*(DF(I,K)-NC*NC*F(I,K)/RI)/RI)*D(I)
6170 SH=-(D3F(I,K)+D2F(I,K)/RI(NC*NC+1)*DF(I,K)/RI/RI+
     2*NC*NC*F(I,K)/R I/RI/RI-INDE*NC*NC*(1-PO(I))
     *(DF(I,K)/RI/RI-F(I,K)/RI/RI/RI))*D(I)
6180 IF YEPR$="no" THEN RETURN ELSE PRINT
     "i=";I,"k=";K,"ri=";RI:PRINT"f(i,k),dw,bm,sh";
     F(I,K),DW,BM,SH
6190 RETURN
6200 REM Explanatiom of the signs for the varios B.C. ,inc.
     the needed equationlist 95-130
6210 REM   ibcw(i)          meaning    graphic not.
     equations  ------           -------    -----------
     ---------
6220 REM      0               S.S       E.     ^-----
     WR=0,BMR=0
              1             FIXED       E.     |-----
     WR=0,DWR=0
              2           F.FIXED       E.    ||-----
     WR=0,SHR=0
6230 REM      3             E.S.S       E.    K^-----
     BMR=0,SHR-KV(I)*WR=0
              4              FREE       E.      -----
     BMR=0,SHR=0
6240 REM     10           CONTINOUS J.        --- ---
     WRL=0,DWRL=0,BMRL=0,SHRL=0
             11             S.S         J.    ---^---
     WL=0,WR=0,DWRL=0,BMRL=0
             12             FIXED       J.    ---|---
     WL=0,WR=0,DWL=0,DWR=0
6250 REM     13             E.S.S       J.    ---K^--
     WLR=0,DWRL=0,BMRL=0,SHRL-vk*WR=0
6260 REM (in case of a HOLE)
6270 REM     11               S.S     e.in   ^-----
     WR=0,BMR=0
             12             FIXED     e.in |-----
     WR=0,DWR=0
             14         F.FIXED       E.in ||-----
     WR=0,SHR=0
6280 REM     13             E.S.S     e.in K^-----
     BMR=0,SHR-vk(I)*WR=0
             10              FREE     e.in   -----
     BMR=0,SHR=0
6290 REM Sub. which checks if the solution satisfies the
     differential equation
```

```
6300 ' VAR:ri,i,k,sofd(i),,sofcd(i),f(i,k)df(i,k),
     d2f(i,k),d3f(i,k), d4f(i  ,k)
6310   DE4=RI*RI*RI*RI*D4F(I,K):DE3=2*RI*RI*RI*D3F(I,K):DE2=-
     RI*RI*(1+2*N C*NC+SOCFD(I)-RI*RI*SOFD(I))*D2F(I,K)
6320   DE1=RI*(1+2*NC*NC+SOCFD(I)+RI*RI*SOFD(I))*DF(I,K)-
     (NC*NC*(4-NC*NC+SOCFD(I)+RI*RI*SOFD(I)))*F(I,K):
     DDDD=DE1+DE2+DE3+DE4:IF DDDD>.0001 THEN GOTO 6340
6330 PRINT:PRINT "i=";I;TAB(20);"k=";K;TAB(40);
     "de(f(";I;",";K;")=";DE1+DE2+DE 3+DE4 :PRINT:PRINT
     "de(f";I;",";K;")=";DE1+DE2+DE3+DE4:RETURN
6340 PRINT:PRINT "i=";I;TAB(20);"k=";K;TAB(40);
     "da(f(";I;",";K;")=";DE1+DE2+DE 3+DE4
     :PRINT:BEEP:BEEP:PRINT "<<<<<< you have a mistake
     >>>>>>>>>":BEEP:RETURN:STOP
6350 REM printout of then results (PRIMARY and BUCKLING)
6360  REM OPEN "b:out.dat" FOR OUTPUT AS #1
6370  REM PE(0)=1:E(2)=1:NPL=2:RES(0)=0:RES(1)=2:
     IBCW(0)=1:IBCW(1)=10:I BCWG$ (0)=" ^":IBCWG$(1)="
     ":PE(1)=1
6380 BEEP:BEEP:BEEP:BEEP
6390 PRINT:PRINT:PRINT"          <<<<<<<<<  RESULT
     >>>>>>>>>>>>>>>":PRINT:PRINT
6400 PRINT #1,:PRINT #1,:PRINT #1,"        <<<<<<<<<  RESULT
     >>>>>>>>>>>>>>>":PRINT #1,:PRINT #1,
6410 PRINT:PRINT" in-plane b.c. ";
6420 PRINT #1,:PRINT #1,"   in-plane b.c. ";
6430 S1$="-----":S2$="=====":SM$="  ":SF$=" |":SE$="
     ":SUM$="":SUMR$=""
6440 SDE$="     ":SUMLF$="":SF1$="->":SF2$="<-":SUMRF$=""
6450 IF E(NPL)=0 THEN NPLMAX=NPL-1 ELSE NPLMAX=NPL
6460 FOR I=1 TO NPLMAX
6470 IF IRES(I-1)=0 THEN SUM$=SUM$+SE$ ELSE IF IRES(I-1)=1
     THEN SUM$=SUM$+SF$ ELSE SUM$=SUM$+SM$
6480 IF PE(I-1)=0 THEN SUMLF$=SUMLF$+SE$ ELSE IF PE(I-1)>0
     THEN SUMLF$=SUMLF$+SF1$ ELSE SUMLF$=SUMLF$+SF2$
6490 SUMLF$=SUMLF$+SDE$
6500 IF I/2-INT(I/2)=0 THEN SUM$=SUM$+S2$ ELSE SUM$=SUM$+S1$
6510 NEXT I
6520 IF E(NPL)=0 THEN GOTO 6530 ELSE GOTO 6550
6530 I=NPL:IF IRES(I-1)=0 THEN SUM$=SUM$+SE$+SDE$ ELSE IF
     IRES(I-1)=1 THEN SUM$=SUM$+SF$+SDE$ ELSE
     SUM$=SUM$+SM$+SDE$
6540 IF PE(I-1)=0 THEN SUMLF$=SUMLF$+SE$+SDE$ ELSE IF PE(I-
     1)>0 THEN SUMLF$=SUMLF$+SF1$+SDE$ ELSE
     SUMLF$=SUMLF$+SF2$+SDE$
6550 IL=LEN(SUM$):FOR I=IL TO 1 STEP -1:
     SUMR$=SUMR$+MID$(SUM$,I,1)
6560 SFF$=MID$(SUMLF$,I,1):IF SFF$=">" THEN SUMRF$=SUMRF$+"<"
     ELSE IF SFF$="<" THEN SUMRF$=SUMRF$+">" ELSE
     SUMRF$=SUMRF$+SFF$
6570 NEXT I
6580 PRINT TAB(20);SUM$+SUMR$:PRINT :PRINT " Forces:
     ";TAB(20);SUMLF$+SUMRF$
```

```
6590 PRINT #1,TAB(20);SUM$+SUMR$:PRINT #1,:PRINT #1," Forces:
     "; TAB(20);SUMLF$+SUMRF$
6600 PRINT:PRINT" Vertical B.C  ";
6610 PRINT #1,:PRINT #1," Vertical B.C  ";
6620 S1$="-----":S2$="=====":SUM$="":SUMR$=""
6630 SDE$="      ":SUMLJ$="":SUMLM$="
     ":SUMRJ$="":SUMRM$="":SDE2$="  "
6640 IF E(NPL)=0 THEN NPLMAX=NPL-1 ELSE NPLMAX=NPL
6650 FOR I=1 TO NPLMAX
6660 SUM$=SUM$+IBCWG$(I-1)
6670 SUMLJ$=SUMLJ$+STR$(I-1)+SDE$:SUMLM$=SUMLM$+STR$(I):IF
     I<NPLMAX THEN SUMLM$=SUMLM$+SDE$  ELSE
     SUMLM$=SUMLM$+SDE2$
6680 IF I/2-INT(I/2)=0 THEN SUM$=SUM$+S2$ ELSE SUM$=SUM$+S1$
6690 NEXT I
6700 IF E(NPL)=0 THEN GOTO 6710 ELSE GOTO 6720
6710 I=NPL:SUM$=SUM$+IBCWG$(I-1)+"
     ":SUMLJ$=SUMLJ$+STR$(I-1)+SDE$:SUMLM$=SUMLM$+"        "
6720 IL=LEN(SUM$):FOR I=IL TO 1 STEP -
     1:SUMR$=SUMR$+MID$(SUM$,I,1)
6730 SUMRJ$=SUMRJ$+MID$(SUMLJ$,I,1):NEXT I
6740 IL=LEN(SUMLM$):FOR I=IL TO 1 STEP -
     1:SUMRM$=SUMRM$+MID$(SUMLM$,I,1):NEXT I
6750 PRINT TAB(20);SUM$+SUMR$:PRINT
     TAB(20);SUMLJ$+SUMRJ$:PRINT  TAB(20);SUMLM$+SUMRM$
6760 PRINT #1,TAB(20);SUM$+SUMR$:PRINT
     #1,TAB(20);SUMLJ$+SUMRJ$:PRINT #1,TAB(20);SUMLM$+SUMRM$
6770 IF IPRI$="no" THEN GOTO 6780 ELSE IPRI$="no":RETURN
6780 PRINT:PRINT:PRINT" <<<<<<<<<  data and results
     >>>>>>>>>>>>":PRINT
6790 PRINT #1,:PRINT #1,:PRINT #1," <<<<<<<<<  data and
     results  >>>>>>>>>>>>":PRINT #1,
6800 PRINT:PRINT TAB(15);" <<<<<        ";NAMEC$;"
     >>>>>>":PRINT
6810 PRINT#1,:PRINT #1,TAB(15);" <<<<<        ";NAMEC$;"
     >>>>>>":PRINT #1,
6820 SS3$="###.###":SS2$="##.##"
6830 PRINT TAB(30);"nc(no. of mode)="; NC
6840 PRINT TAB(30);"po(poison rat.)=";:PRINT USING SS2$;POO
6850 PRINT TAB(30);"th(thickness  )=";:PRINT USING SS2$;T(1)
6860 PRINT #1,TAB(30);"nc(no. of mode)=";NC
6870 PRINT #1,TAB(30);"po(poison rat.)=";:PRINT #1,USING
     SS2$;POO
6880 PRINT #1,TAB(30);"th(thickness  )=";:PRINT #1,USING
     SS2$;T(1)
6890 PRINT :PRINT :PRINT TAB(20);" Buckling Force (Pcr*D/R/R)
     ":PRINT TAB(19);"---------------------------------
     ":PRINT
6900 PRINT #1,:PRINT #1,:PRINT #1,TAB(20);" Buckling Force
     (Pcr*D/R/R)  ":PRINT #1,TAB(19);"----------------------
     ----------":PRINT #1,
6910 IF IYEINTE=1 THEN IYEINTE=0:RETURN
```

```
6920 PRINT TAB(1);"r1/r0 \e2/e1";TAB(15);:PRINT USING
     SS3$;E22(1);:PRINT TAB(25);:PRINT USING
     SS3$;E22(2);:PRINT TAB(35);:PRINT USING
     SS3$;E22(3);:PRINT TAB(45);:PRINT USING SS3$;E22(4);
6930 PRINT TAB(55);:PRINT USING SS3$;E22(5);:PRINT
     TAB(65);:PRINT USING SS3$;E22(6);:PRINT TAB(75);:PRINT
     USING SS3$;E22(7):PRINT
6940 FOR IR10=1 TO 9:RRI=IR10/10:PRINT TAB(4);:PRINT USING
     SS2$;RRI;
6950 FOR IE21=1 TO 7:PRINT TAB(3+IE21*10);:PRINT USING
     SS3$;PCRR(IE21,IR10)*R(0)*R(0);
6960 NEXT IE21:NEXT IR10
6970 PRINT #1,TAB(1);"r1/r0 \e2/e1";TAB(15);:PRINT #1,USING
     SS3$;E22(1);:PRINT  #1,TAB(25);:PRINT #1,USING
     SS3$;E22(2);:PRINT #1,TAB(35);:PRINT #1,USING
     SS3$;E22(3);:PRINT #1,TAB(45);:PRINT #1,USING
     SS3$;E22(4);
6980 PRINT #1,TAB(55);:PRINT #1,USING SS3$;E22(5);:PRINT
     #1,TAB(65);:PRINT #1,USING SS3$;E22(6);:PRINT
     #1,TAB(75);:PRINT #1,USING SS3$;E22(7):PRINT
6990 FOR IR10=0 TO 9:RRI=IR10/10:PRINT #1,TAB(4);:PRINT
     #1,USING SS2$;RRI;
7000 FOR IE21=1 TO 7:PRINT #1,TAB(3+IE21*10);:PRINT #1,USING
     SS3$;PCRR(IE21,IR10)*R(0)*R(0);
7010 NEXT IE21:NEXT IR10
7020 RETURN
7030 REM Sub. for printout of normal mode
7040 SS1$="##.##":SS2$="###.###":SS3$="####.###"
7050 PRINT :PRINT TAB(25);"   Normal Mode #";NC;"    "
7060 PRINT TAB(25);"----------------------"
7070 PRINT:PRINT:PRINT
     TAB(28);"Pcr=";PCRR(IR10,IE21)*R(0)*R(0)/1;" D0/r(0)^2"
7080 PCRR1=PCRR(IR10,IE21):PRINT:PRINT
7090 PRINT TAB(3);"rr";TAB(14);"Nrr";TAB(24);"Noo";
     TAB(33);"U(Hor.)";TAB (43);
     "W(Ver.)";TAB(53);"Dw/dr";TAB(63);"B.mom";
     TAB(73);"Shear":PRI NT
7100 FOR I=1 TO NPLI:FOR J=0 TO 10
7110  PRINT TAB(1);:PRINT USING SS1$;RR(I,J)/R(0);:PRINT
     TAB(10);:PRINT USING SS2$;PCRR1*SRR(I,J);:PRINT
     TAB(20);:PRINT USING SS2$;SOO(I,J)*PCRR1;:PRINT
     TAB(30);:PRINT USING SS2$;UU(I,J)*PCRR1;
7120  PRINT TAB(40);:PRINT USING SS3$;WW(I,J)/WMAX;:PRINT
     TAB(50);:PRINT USING  SS3$;DWW(I,J)/WMAX;:PRINT
     TAB(60);:PRINT USING SS3$;BMW(I,J)/WMAX;:
     PRINT TAB(70);:PRINT USING SS3$;SHW(I,J)/WMAX:NEXT
     J:NEXT I
7130 IF E(NPL)=0 THEN GOTO 7160 ELSE I=NPL:FOR J=0 TO 5
7140  PRINT TAB(1);:PRINT USING SS1$;RR(I,J)/R(0);:PRINT
     TAB(10);:PRINT USING SS2$;PCRR1*SRR(I,J);:PRINT
     TAB(20);:PRINT USING SS2$;SOO(I,J)*PCRR1;:PRINT
     TAB(30);:PRINT USING SS2$;UU(I,J)*PCRR1;
```

```
7150  PRINT TAB(40);:PRINT USING SS3$;WW(I,J)/WMAX;:PRINT
     TAB(50);:PRINT USING  SS3$;DWW(I,J)/WMAX;:PRINT
     TAB(60);:PRINT USING SS3$;BMW(I,J)/WMAX;:
     PRINT TAB(70);:PRINT USING SS3$;SHW(I,J)/WMAX:NEXT J
7160 PRINT #1,:PRINT #1,TAB(25);"   Normal Mode #";NC;"     "
7170 PRINT #1,TAB(25);"----------------------"
7180 PRINT #1,:PRINT #1,:PRINT
     #1,TAB(28);"Pcr=";PCRR(IR10,IE21)*R(0)*R(0)/1;"
     DO/r(0)^2"
7190 PRINT #1,:PRINT #1,
7200 PRINT #1,TAB(3);"rr";TAB(14);"Nrr";TAB(24);"Noo";
     TAB(33);"U(Hor.)"; TAB(43);"W(Ver.)";TAB(53);
     "Dw/dr";TAB(63);"B.mom";TAB(73);"Shear":PRINT #1,
7210 FOR I=1 TO NPLI:FOR J=0 TO 10
7220  PRINT #1,TAB(1);:PRINT #1,USING
     SS1$;RR(I,J)/R(0);:PRINT #1,TAB(10);:PRINT #1,USING
     SS2$;PCRR1*SRR(I,J);:PRINT #1,TAB(20);:PRINT #1,USING
     SS2$;SOO(I,J)*PCRR1;:PRINT #1,TAB(30);:PRINT #1,USING
     SS2$;UU(I,J)*PCRR1;
7230  PRINT #1,TAB(40);:PRINT #1,USING
     SS3$;WW(I,J)/WMAX;:PRINT #1,TAB(50);:PRINT #1,USING
     SS3$;DWW(I,J)/WMAX;:PRINT #1,TAB(60);:PRINT #1,USING
     SS3$;BMW(I,J)/WMAX;:PRINT #1,TAB(70);:PRINT #1,USING
     SS3$;SHW(I,J)/WMAX:NEXT J:NEXT I
7240 IF E(NPL)=0 THEN GOTO 7270 ELSE I=NPL:FOR J=0 TO 5
7250  PRINT #1,TAB(1);:PRINT #1,USING
     SS1$;RR(I,J)/R(0);:PRINT #1,TAB(10);:PRINT #1,USING
     SS2$;PCRR1*SRR(I,J);:PRINT #1,TAB(20);:PRINT #1,USING
     SS2$;SOO(I,J)*PCRR1;:PRINT #1,TAB(30);:PRINT #1,USING
     SS2$;UU(I,J)*PCRR1;
7260  PRINT #1,TAB(40);:PRINT #1,USING
     SS3$;WW(I,J)/WMAX;:PRINT #1,TAB(50);:PRINT #1,USING
     SS3$;DWW(I,J)/WMAX;:PRINT #1,TAB(60);:PRINT #1,USING
     SS3$;BMW(I,J)/WMAX;:PRINT #1,TAB(70);:PRINT #1,USING
     SS3$;SHW(I,J)/WMAX:NEXT J
7270 RETURN
```